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The Moduli Space of Boundary Compactifications of $\mathrm{SL}(2,\mathbb{R})$

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Abstract. In an earlier paper, the authors introduced the notion of a *boundary compactification* of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, a normal projective embedding of PSL_2 arising as the Zariski closure of an orbit in $(\mathbb{P}^1)^n$ under the diagonal action of SL_2. Here the moduli space of such boundary compactifications of $\mathrm{SL}(2,\mathbb{R})$ is shown to be a contractible hyperbolic orbifold, by using the Schwarz–Christoffel transformation to identify it with a quotient of the moduli space of equi-angular planar polygons.

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1. Introduction

If G is an algebraic group over an algebraically closed field and H is an algebraic subgroup, a *(normal) embedding* of G/H is an irreducible (normal) algebraic variety with an algebraic action of G and which contains G/H as an open orbit. In the early 1980's Luna and Vust [L-V], taking inspiration from earlier work of Popov [Pop], completed a classification of normal embeddings of G/H (where now G is a reductive group and a Borel subgroup of G has an orbit of codimension less than or equal to one) which resembles the classification of toral embeddings. For the particular case $G = \mathrm{SL}(2,\mathbb{C})$ and $H = \{\mathrm{Id}\}$, they carry out the explicit calculation and give a characterization of each embedding in terms of a diagram which encodes combinatorial data for the local rings of the orbits. Then L. Moser-Jauslin in the late 1980's extended the diagram characterization to the case in which $G = \mathrm{SL}(2,\mathbb{C})$ and H is any finite subgroup ([MJ1] and [MJ2]) and also translated geometric properties of the embeddings into numerical conditions on the data in the diagrams.

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In [I-Por1] we give a geometric interpretation of some of these embeddings. In particular, we are interested only in the embeddings into projective varieties and the case in which $H = \{\pm \mathrm{Id}\}$. However, our description applies to both of the cases $G = \mathrm{SL}(2, \mathbb{C})$ and $G = \mathrm{SL}(2, \mathbb{R})$, and uses in an essential way the geometry at infinity of the associated symmetric space. Hence, as remarked in the introduction of [I-Por1], it seems to have a good chance of being extended at least to other groups of $\mathbb{R}$-rank 1, and perhaps even further.

DEFINITION 1.1 [I-Por1]. Let G be a connected k-algebraic group. An irreducible normal projective (smooth) k-algebraic variety X on which G acts algebraically is a *(smooth) algebraic compactification* of G if $\dim X = \dim G$ and X has a G-orbit with finite stabilizer.

The algebraic compactifications with which we shall deal in this paper arise from the diagonal action of $G = \mathrm{SL}(2, \mathbb{R})$ on the product of copies of the boundary of the real hyperbolic plane.

DEFINITION 1.2 [I-Por1]. Let $\mathcal{X}$ be the boundary of real hyperbolic two-space with the usual $\mathrm{SL}(2, \mathbb{R})$-action and let $\mathrm{SL}(2, \mathbb{R})$ act diagonally on $\mathcal{X}^n$. A *boundary compactification of* $\mathrm{SL}(2, \mathbb{R})$ is the Zariski closure X of the orbit of a point $p \in \mathcal{X}^n$ whose stabilizer is discrete. The integer n is the *embedding dimension* of X.

We showed in [I-Por1] that for each integer $n \geqslant 3$ there is an Zariski open set $\mathcal{D}^{n-3} \subset \mathbb{R}^{n-3}$ which parameterizes the boundary compactifications of $\mathrm{SL}(2, \mathbb{R})$ of embedding dimension n, and while on the one hand we were interested in relating this construction to the classification of Luna, Vust and Moser-Jauslin, on the other hand we also wanted to understand to what extent the G-homeomorphism classes of these boundary compactifications are distinct. With regards to the first issue, we prove in [I-Por1] that our boundary compactifications are projective normal embeddings which correspond to one of the kinds of combinatorial diagrams in [MJ2]. For the second issue, let us make the following clarification:

DEFINITION 1.3. Two algebraic compactifications X_1 and X_2 of G are *topologically G-equivalent* if there is a homeomorphism $X_1 \to X_2$ which commutes with the G-action.

It is quite obvious that any permutation in the coordinates of a boundary compactification is a continuous map which commutes with the action of $\mathrm{SL}(2, \mathbb{R})$; the somewhat surprising result is that these are really the only topological $\mathrm{SL}(2, \mathbb{R})$-isomorphisms between boundary compactifications. (For a more precise statement of this result see [I-Por1] or Theorem 2.3, below.) What one infers from this is that there is an action of the symmetric group S_n on n letters on $\mathcal{D}^{n-3}$ and this paper is devoted to the study of the corresponding space of deformations $\mathcal{M}_n = \mathcal{D}^{n-3}/S_n$.

Our approach to this investigation is in two steps. First, we relate the moduli space $\mathcal{M}_n$ to the moduli space of equi-angular planar polygons up to Euclidean motions and rescaling. We do this using the Schwarz-Christoffel transformation, a conformal map which relates ordered n-tuples of points on the unit circle up to $\mathrm{SL}(2,\mathbb{R})$-transformations with n-gons in the Euclidean plane, up to translations, rotations and scaling. Then using results of Bavard and Ghys [Ba-Gh], we obtain (Section 4) the following theorem.

THEOREM. *$\mathcal{M}_n$ is the quotient of a hyperbolic polyhedron $\mathcal{H}_n^{\mathrm{eq}}$ with totally geodesic faces by an isometric action of the dihedral group $\mathbb{D}_n$ with a fixed point. $\mathcal{M}_n$ has finite volume if and only if $n > 4$ and is a contractible space for every n.*

Note that while our work here applies only to the case $G = \mathrm{SL}(2,\mathbb{R})$, we believe that a thorough understanding of Thurston's work [T] will allow us to use a similar method to study the moduli space of boundary compactifications of $\mathrm{SL}(2,\mathbb{C})$ as well.

2. Background

We recall in this section the construction of boundary compactifications which was carried out in [I-Por1], beginning with some notation. As anticipated in the introduction, let $\mathcal{X}$ denote the boundary of real hyperbolic two-space with orientation preserving isometry group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\,\mathrm{Id}\}$ and full isometry group the general Möbius group $\mathrm{GM}(\mathbb{R})$. Let $\mathrm{SL}(2,\mathbb{R})$ act on $\mathcal{X}^n$ diagonally (where the action on $\mathcal{X}$ is the natural boundary action) and define the following $\mathrm{SL}(2,\mathbb{R})$-invariant sets, which we shall use extensively in the sequel:

(1) $\Delta_0^n = \{(z, z, \dots, z) \in \mathcal{X}^n\} \simeq \mathcal{X}$;
(2) $\Delta_j^n = \{(z_1, z_2, \dots, z_n) \in \mathcal{X}^n : z_l = z_m$, whenever $l, m \neq j$, while $z_i \neq z_j$ for $i \neq j\} \simeq \mathcal{X}^2 \backslash \Delta_0^2$, e.g., $\Delta_1^n = \{(z, w, \dots, w) \in \mathcal{X}^n : z \neq w\}$;
(3) $\Delta_{ij}^n = \{(z_1, z_2, \dots, z_n) \in \mathcal{X}^n : z_l = z_m$ whenever $\{l, m\} \neq \{i, j\}$, while $z_j \neq z_l \neq z_i \neq z_j$ whenever $l \neq i, j\}$, e.g., $\Delta_{12}^n = \{(z_1, z_2, w, \dots, w) \in \mathcal{X}^n : z_1 \neq z_2 \neq w \neq z_1\}$;
(4) $D^n = \{(z_1, z_2, \dots, z_n) \in \mathcal{X}^n : z_i \neq z_j$ if $i \neq j\}$.

(Here $\simeq$ means isomorphism as algebraic $\mathrm{SL}(2,\mathbb{R})$-spaces.)

If $n = 3$ we have a decomposition of $\mathcal{X}^3 \setminus D^3$ into $\mathrm{SL}(2,\mathbb{R})$-orbits as follows

$$\mathcal{X}^3 \backslash D^3 = (\Delta_1^3 \cup \Delta_2^3 \cup \Delta_3^3) \cup \Delta_0^3,$$

where for $j = 1, 2, 3$, $\Delta_j^3 \simeq \mathrm{SL}(2,\mathbb{R})/A$ and $\Delta_0^3 \simeq \mathrm{SL}(2,\mathbb{R})/P$, with A the subgroup of $\mathrm{SL}(2,\mathbb{R})$ of diagonal matrices and P the parabolic subgroup of $\mathrm{SL}(2,\mathbb{R})$ consisting of upper triangular matrices. Notice that because $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{X}$ by only orientation-preserving isometries, D^3 also decomposes into two $\mathrm{SL}(2,\mathbb{R})$-orbits, corresponding to the two different cyclic orientations.

When $n > 3$, D^n is foliated by orbits of $\mathrm{GM}(\mathbb{R})$ (or pairs of $\mathrm{SL}(2,\mathbb{R})$-orbits), which are parameterized by a generalized cross ratio. For this, define $c_n\colon D^n \to \mathbb{R}^{n-3}$ by

$$c_n(z_1, z_2, w_1, \ldots, w_{n-3}, z_3) = (c(z_1, z_2, w_1, z_3), \ldots, c(z_1, z_2, w_{n-3}, z_3)),$$

where

$$c(x, y, z, w) = \frac{(x-z)(y-w)}{(x-y)(z-w)}$$

is the classical cross-ratio. Then the following properties are easy to verify.

LEMMA 2.1 [I-Por1]. (i) *Given* $p \in D^n$, $c_n(p) = (t_1, \ldots, t_{n-3})$ *if and only if there exists* $g \in \mathrm{GM}(\mathbb{R})$ *such that*

$$g \cdot p = (0, 1, t_1, \ldots, t_{n-3}, \infty).$$

(ii) c_n *surjects onto the set*

$$\mathcal{D}^{n-3} = \{(t_1, \ldots, t_{n-3}) \in (\widehat{\mathbb{R}} \backslash \{0, 1, \infty\})^{n-3} : t_i \neq t_j,\ \textit{if}\ i \neq j\}.$$

(iii) *For every* $t \in \mathcal{D}^{n-3}$, $D^n_t = c_n^{-1}(t) \subset D^n$ *is a* $\mathrm{GM}(\mathbb{R})$*-orbit which decomposes into two* $\mathrm{SL}(2,\mathbb{R})$*-orbits* $D^n_{t,+}$ *and* $D^n_{t,-}$ *(corresponding to different orientations), each with stabilizer* $\{\pm \mathrm{Id}\}$.

Using the fixed-point properties of the actions on $\mathcal{X}$ of the various elliptic, parabolic and hyperbolic elements of $\mathrm{SL}(2,\mathbb{R})$, one can see that for any $p \in D^n$ with $c_n(p) = t \in \mathcal{D}^{n-3}$, $\mathrm{GM}(\mathbb{R}) \cdot p = c_n^{-1}(t)$ and

$$\overline{c_n^{-1}(t)} \smallsetminus c_n^{-1}(t) = \left(\bigcup_{j=1}^{n} \Delta^n_j \right) \cup \Delta^n_0.$$

(Here and below we take closures with respect to the Hausdorff topology.) Note that, since the right-hand side is independent of $t \in \mathcal{D}^{n-3}$, we have that

$$\bigcap_{t \in \mathcal{D}^{n-3}} \overline{c_n^{-1}(t)} = \left(\bigcup_{j=1}^{n} \Delta^n_j \right) \cup \Delta^n_0.$$

Then one can show the following theorem:

THEOREM 2.2 [I-Por1]. *For any* $n > 3$ *and* $t \in \mathcal{D}^{n-3}$, *the set* $\overline{c_n^{-1}(t)} = X^n_t$ *is an algebraic compactification of* $\mathrm{SL}(2,\mathbb{R})$ *whose singular set is* Δ^n_0.

To study the moduli space of these boundary compactifications, it is necessary to see when there can be $\mathrm{SL}(2,\mathbb{R})$-homeomorphisms among them. First one identifies the possible $\mathrm{SL}(2,\mathbb{R})$-homeomorphisms between closures of $\mathrm{SL}(2,\mathbb{R})$-orbits by examining the limits of orbits of certain one-parameter subgroups. Then when working with a boundary compactification, the separate results on the two $\mathrm{SL}(2,\mathbb{R})$-orbit closures can be compared by examining their behavior on the common boundaries of these $\mathrm{SL}(2,\mathbb{R})$-orbits. The final result is the following:

THEOREM 2.3 [I-Por1]. *Let $t, t' \in \mathcal{D}^{n-3}$, $s \in \mathcal{D}^{m-3}$, with $n \neq m$. Then*

(1) *X_t^n and X_s^m are never topologically $\mathrm{SL}(2,\mathbb{R})$-isomorphic;*
(2) *A map $\varphi\colon X_t^n \to X_s^m$ is a topological $\mathrm{SL}(2,\mathbb{R})$-isomorphism if and only if it is a permutation $\sigma \in S_n$.*

It thus makes sense to define an action of S_n on $\mathcal{D}^{n-3}$ as follows. For $\sigma \in S_n$ and $(t_1,\dots,t_{n-3}) \in \mathcal{D}^{n-3}$, set

$$\sigma(t_1,\dots,t_{n-3}) = c_n(\sigma(0,1,t_1,\dots,t_{n-3},\infty)).$$

Denote by $\mathcal{M}_n$ the set of topological $\mathrm{SL}(2,\mathbb{R})$-isomorphism classes of boundary compactifications with embedding dimension n.

COROLLARY 2.4 [I-Por1]. *$\mathcal{M}_n$ is in natural bijective correspondence with $S_n \backslash \mathcal{D}^{n-3}$.*

While in a forthcoming paper [I-Por2] we shall study $\mathcal{M}_n$ using the S_n-action directly, in this paper we bypass this route with the help of the Schwarz–Christoffel transformation, as explained in the next section.

3. The Schwarz–Christoffel Transformation

We describe now a basic tool which will allow us to relate the moduli space $\mathcal{M}_n$ with the moduli space of appropriate plane polygons up to homotheties. In fact, the Schwarz–Christoffel transformation, which we shall describe in this section, is merely a realization of the biholomorphism between the unit disc $\mathcal{U}$ in the complex plane and the interior of a given polygonal domain, whose existence is guaranteed by the Riemann mapping theorem (see Figure 1).

The classical definition of this transformation is as follows. Let $z_1,\dots,z_n \in S^1$ be an n-tuple of unit complex numbers in positive cyclic order and let $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ be constants such that $\sum_{j=1}^{n} \alpha_j = 2$ and $-1 < \alpha_j < 1$ for $1 \leqslant j \leqslant n$. For any constants $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, the *Schwarz–Christoffel transformation* is the map f defined on $\overline{\mathcal{U}}$ by

$$f(z) = \alpha \int_0^z \frac{\mathrm{d}w}{(w-z_1)^{\alpha_1}\dots(w-z_n)^{\alpha_n}} + \beta,$$

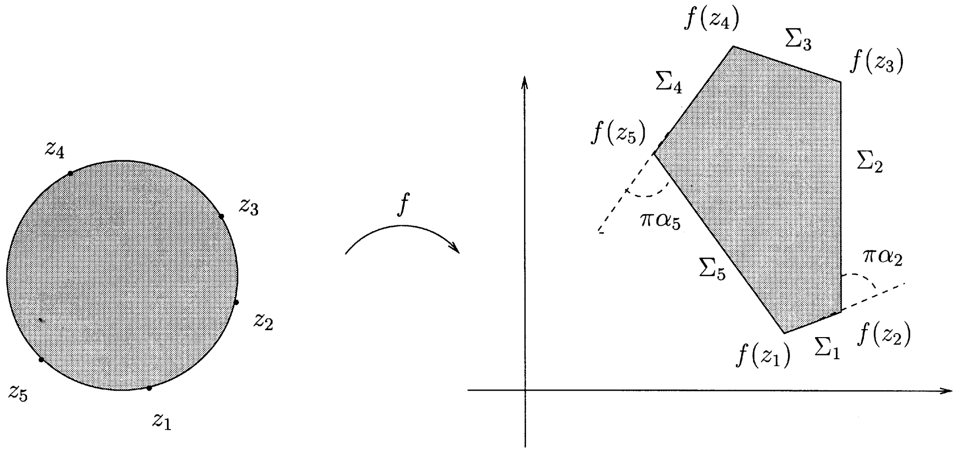

Figure 1. The Schwarz–Christoffel transformation.

where the integration is along any path joining 0 to z and we choose branches for the powers in the integrand with branch cuts that avoid $\mathcal{U}$. If we set

$$\ell_j = \int_{z_{j-1}}^{z_j} \left| \frac{\mathrm{d}w}{(w - z_1)^{\alpha_1} \ldots (w - z_n)^{\alpha_n}} \right|,$$

then it is not hard to show that f maps the unit circle onto a closed polygonal line $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$, whose sides Σ_j are the straight segments $\Sigma_j = [f(z_j), f(z_{j+1})]$ of length ℓ_j and forming an angle $\alpha_{j+1}\pi$ with the following side Σ_{j+1} (here indices are to be treated modulo n). Hence, the jth interior angle of the polygon Σ is equal to $(1 - \alpha_j)\pi$.

Moreover, if we assume that the z_j's and α_j's are such that the polygon Σ is a Jordan curve, then the Schwarz–Christoffel transformation is a homeomorphism of $\overline{\mathcal{U}}$ onto $\overline{\mathrm{Int}(\Sigma)}$ and a biholomorphism of $\mathcal{U}$ onto $\mathrm{Int}(\Sigma)$. Conversely, for any polygon forming a Jordan curve in the plane with exterior angles $\alpha_1\pi, \ldots, \alpha_n\pi$, there exists an n-tuple of points on S^1 in positive cyclic order such that the corresponding Schwarz–Christoffel transformation has the given polygon as image. Moreover, up to a choice of the complex constants α and β, the Schwarz–Christoffel transformation is the unique map with the above properties. (See, for instance, [Be-Ga] or [N] for a proof of these facts.)

For our purpose, it will be enough to specialize the above construction to the case $\alpha_j = 2/n$ for $j = 1, \ldots, n$. We shall write $\mathcal{P}_n^{\mathrm{eq}}$ for the set of labeled convex n-gons in $\mathbb{C}$ with all exterior angles equal to $2\pi/n$ and D_+^n for the set of n-tuples $(z_1, \ldots, z_n) \in D^n$ such that the $z_1, \ldots, z_n$ are in positive cyclic order on S^1. We define the map $\mathcal{SC}\colon D_+^n \to \mathcal{P}_n^{\mathrm{eq}}$ by setting $\mathcal{SC}(z_1, \ldots, z_n)$ to be the equi-angular n-gon which is the image of the map

$$f_{(z_1,\ldots,z_n)}(z) = \int_0^z \frac{\mathrm{d}w}{(w - z_1)^{2/n} \ldots (w - z_n)^{2/n}}$$

(in other words, the image of the Schwarz–Christoffel transformation with α and β fixed to be 1 and 0, respectively), with the obvious labeling. We want to describe now how the action of $\mathrm{SL}(2,\mathbb{R})$ on D_+^n affects the image of the map $\mathcal{SC}$.

Let $\mathcal{A}$ be the group $\mathbb{C}^* \ltimes \mathbb{C}$ acting on the complex plane by $\rho_{(\alpha,\beta)}z = \alpha z + \beta$ and consider the action of $\mathcal{A}$ on $\mathcal{P}_n^{\mathrm{eq}}$. Two labeled polygons are *similar* if they can be obtained one from the other, in a label-preserving way, by translation, rotation and rescaling, *i.e.*, if they are in the same $\mathcal{A}$-orbit in $\mathcal{P}_n^{\mathrm{eq}}$. The following result is the building block of what follows.

PROPOSITION 3.1. *$\mathcal{SC}(z_1, \ldots, z_n)$ is similar to $\mathcal{SC}(w_1, \ldots, w_n)$ if and only if there exists $g \in \mathrm{PSL}(2,\mathbb{R})$ such that $g \cdot (z_1, \ldots, z_n) = (w_1, \ldots, w_n)$.*

Proof. If $\mathcal{SC}(z_1, \ldots, z_n)$ is similar to $\mathcal{SC}(w_1, \ldots, w_n)$, then there exists $(\alpha, \beta) \in \mathcal{A}$ such that $\rho_{(\alpha,\beta)}(\mathcal{SC}(z_1, \ldots, z_n)) = \mathcal{SC}(w_1, \ldots, w_n)$, preserving the labeling. Thus the composition of the maps $f^{-1}_{(w_1,\ldots,w_n)} \circ \rho_{(\alpha,\beta)} \circ f_{(z_1,\ldots,z_n)}$ is a holomorphic automorphism of $\mathcal{U}$ which extends to a homeomorphism of $\overline{\mathcal{U}}$ sending $(z_1, \ldots, z_n)$ to $(w_1, \ldots, w_n)$. Hence, there exists $g \in \mathrm{PSL}(2,\mathbb{R})$ such that $g \cdot (z_1, \ldots, z_n) = (w_1, \ldots, w_n)$.

Conversely, if $g \cdot (z_1, \ldots, z_n) = (w_1, \ldots, w_n)$ for $g \in \mathrm{PSL}(2,\mathbb{R})$, then $\varphi \colon z \mapsto f_{(w_1,\ldots,w_n)}(gz)$ is a Riemann mapping which sends $\mathcal{U}$ to the interior of an n-gon with all exterior angles equal to $2\pi/n$ and the points $z_1, \ldots, z_n$ to the vertices of this n-gon. By the uniqueness of the Schwarz–Christoffel transformation it follows that there exist $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, such that φ is the Schwarz–Christoffel transformation with constants α and β. But this is exactly the same as saying that $\rho_{(\alpha,\beta)}\mathcal{SC}(z_1, \ldots, z_n) = \alpha\mathcal{SC}(z_1, \ldots, z_n) + \beta = \mathcal{SC}(w_1, \ldots, w_n)$. □

COROLLARY 3.2. *$D_+^n/\mathrm{PSL}(2,\mathbb{R})$ and $\mathcal{P}_n^{\mathrm{eq}}/\mathcal{A}$ are homeomorphic.*

Proof. The homeomorphism is the map $\overline{\mathcal{SC}} : D_+^n/P\mathrm{SL}(2,\mathbb{R}) \to \mathcal{P}_n^{\mathrm{eq}}/\mathcal{A}$ induced by $\mathcal{SC}$, which is well-defined by Proposition 3.1. Since $\mathrm{PSL}(2,\mathbb{R})$ acts freely on D_+^n if $n \geqslant 3$, $\overline{\mathcal{SC}}$ is injective, while surjectivity follows from the properties of the Schwarz–Christoffel transformation. The continuity of $\overline{\mathcal{SC}}$ and its inverse follows from the definition of $\mathcal{SC}$. □

Let us return now to the moduli space of boundary compactifications $\mathcal{M}_n$ and spell out exactly how it relates to the spaces of polygons we have now introduced. Recall that the dihedral group $\mathbb{D}_n \subset S_n$ is the group of symmetries of an n-gon and is isomorphic to $\mathbb{Z}_2 \ltimes \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the subgroup of rotations and $\mathbb{Z}_2$ contains a reflection. $\mathbb{D}_n$ acts on $\mathcal{P}_n^{\mathrm{eq}}$ by changing the labeling of polygons.

THEOREM 3.3. *There is a homeomorphism $\mathcal{M}_n \cong \mathbb{D}_n \backslash \mathcal{P}_n^{\mathrm{eq}}/\mathcal{A}$.*

Proof. Since our boundary compactifications are topological closures of orbits of the general Möbius group, which includes also orientation-reversing transform-

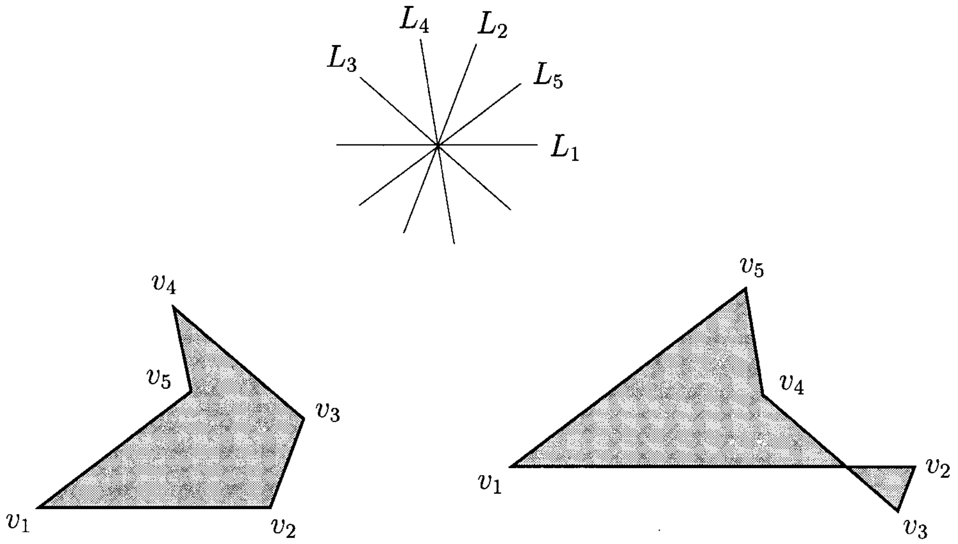

Figure 2. An example of a choice of lines $\mathcal{L}$ together with two polygons in the corresponding $P_n^{\mathcal{L}}$.

ations, we need also to introduce the set $D^n_\pm$ of points $(z_1, \dots, z_n) \in D^n$ which are in positive *or* negative cyclic order. Then from Corollary 3.2 we have

$$D^n_\pm/\mathrm{GM}(\mathbb{R}) = D^n_+/\mathrm{PSL}(2,\mathbb{R}) \cong \mathcal{P}_n^{\mathrm{eq}}/\mathcal{A},$$

and moreover it is straightforward to see that $S_n \backslash D^n = \mathbb{D}_n \backslash D^n_\pm$ (where the action of S_n on D^n is by permutations of the coordinates). Now recall that $\mathcal{D}^{n-3} \cong D^n/\mathrm{GM}(\mathbb{R})$ and that the diagonal action of the general Möbius group on D^n commutes with the action of the symmetric group. Then, by Corollary 2.4 and the results summarized in Section 2, we have

$$\begin{aligned} \mathcal{M}_n &= S_n \backslash \mathcal{D}^{n-3} \cong S_n \backslash \left(D^n/\mathrm{GM}(\mathbb{R})\right) = \left(S_n \backslash D^n\right)/\mathrm{GM}(\mathbb{R}) \\ &= \left(\mathbb{D}_n \backslash D^n_\pm\right)/\mathrm{GM}(\mathbb{R}) = \mathbb{D}_n \backslash \left(D^n_\pm/\mathrm{GM}(\mathbb{R})\right) \cong \mathbb{D}_n \backslash \mathcal{P}_n^{\mathrm{eq}}/\mathcal{A}. \qquad \square \end{aligned}$$

4. The Moduli Space of Polygons à la Thurston–Bavard–Ghys

In this section we shall summarize (mostly without proofs) some of the results of [Ba-Gh] (which were in turn inspired by [T]) and use them to identify the space $\mathcal{P}_n^{\mathrm{eq}}/\mathcal{A}$ and its quotient by $\mathbb{D}_n$.

If $n > 3$, a *centered labeled n-gon* in the plane is an n-tuple $(v_1, \dots, v_n)$ of vectors in $\mathbb{R}^2$ whose sum is zero; the corresponding polygon has vertices at the points v_j with the obvious labeling. We shall write Π_n for the set of all centered labeled n-gons, which is naturally a $(2n-2)$-dimensional real vector space. Given a point $\mathcal{L} = (L_1, \dots, L_n) \in \left(\mathbb{RP}^1\right)^n$, there is an $(n-2)$-dimensional subspace

$P_n^{\mathcal{L}} \subset \Pi_n$ consisting of those polygons for which $v_{j+1} - v_j \in L_j$ for all $1 \leqslant j \leqslant n$ (where, as before, indices are to be interpreted modulo n). Notice that with this definition the sides of the polygons in $P_n^{\mathcal{L}}$ are allowed to cross each other, as in Figure 2.

The subspaces $P_n^{\mathcal{L}}$ have an interesting geometry which comes from the quadratic form $A\colon \Pi_n \to \mathbb{R}$ defined by

$$A(v_1, \ldots, v_n) = \frac{1}{2}\sum_{j=1}^{n} \det(v_j, v_{j+1}).$$

This computes the area of, for example, a convex polygon, but what is particularly useful is that its restriction to $P_n^{\mathcal{L}}$ has a signature which can be computed from the oriented angles between the successive lines of $\mathcal{L}$. The crucial case for us is when $\mathcal{L}$ is such that $P_n^{\mathcal{L}}$ contains convex polygons, in which case we shall call $\mathcal{L}$ *convex*.

PROPOSITION 4.1 [Ba-Gh]. *If $\mathcal{L}$ is convex, then the signature of A is $(1, n-3)$. Hence, for such $\mathcal{L}$, the projectivization of the positive cone of A is the Klein model of real hyperbolic space of dimension $n-3$.*

It is useful to look at certain subsets of $P_n^{\mathcal{L}}$ corresponding to a choice of orientation for each of the lines L_j in $\mathcal{L}$. We can encode these orientations by writing $\mathbb{L} = (L_1', \ldots, L_n')$ for a n-tuple of closed half-lines in $\mathbb{R}^2$, and we then define the closed cone

$$\overline{\mathcal{C}}_n^{\mathbb{L}} = \{(v_1, \ldots, v_n) \in P_n^{\mathcal{L}} : v_{j+1} - v_j \in L_j', \text{ for } j = 1, \ldots, n\} \subset P_n^{\mathcal{L}}$$

and its interior $\mathcal{C}_n^{\mathbb{L}}$, consisting of polygons in $\overline{\mathcal{C}}_n^{\mathbb{L}}$ with all vertices distinct. If one of these cones $\overline{\mathcal{C}}_n^{\mathbb{L}}$ contains a single convex polygon, then it consists entirely of convex polygons, and we shall again call the corresponding $\mathbb{L}$ *convex*.

The codimension one faces of $\overline{\mathcal{C}}_n^{\mathbb{L}}$ consist of n-gons with one degenerate side, *i.e.*, $v_j = v_{j+1}$ for some j. Hence, these faces consist of the intersection with $\overline{\mathcal{C}}_n^{\mathbb{L}}$ of a subspace of $P_n^{\mathcal{L}}$ of codimension one. The limit points at infinity of $\mathcal{C}_n^{\mathbb{L}}$ consist of those n-gons which have degenerated to have zero area, which can happen exactly when two of the lines of $\mathcal{L}$ are parallel. Figure 3 shows two points π_1 and π_2 in one $\mathcal{C}_n^{\mathbb{L}}$, a point π_3 on the boundary of the corresponding $\overline{\mathcal{C}}_n^{\mathbb{L}}$ and a point π_4 in the positive cone of A on $P_n^{\mathcal{L}}$ but not in $\overline{\mathcal{C}}_n^{\mathbb{L}}$.

We summarize some of the geometry of these cones in the following

PROPOSITION 4.2 [Ba-Gh]. *For convex $\mathbb{L}$, the projectivization $\mathbb{P}\overline{\mathcal{C}}_n^{\mathbb{L}}$ is a polyhedron with totally-geodesic faces and finitely many limit points at infinity, corresponding to pairs of parallel lines in $\mathbb{L}$ and, hence, is always of finite volume if $n > 4$.*

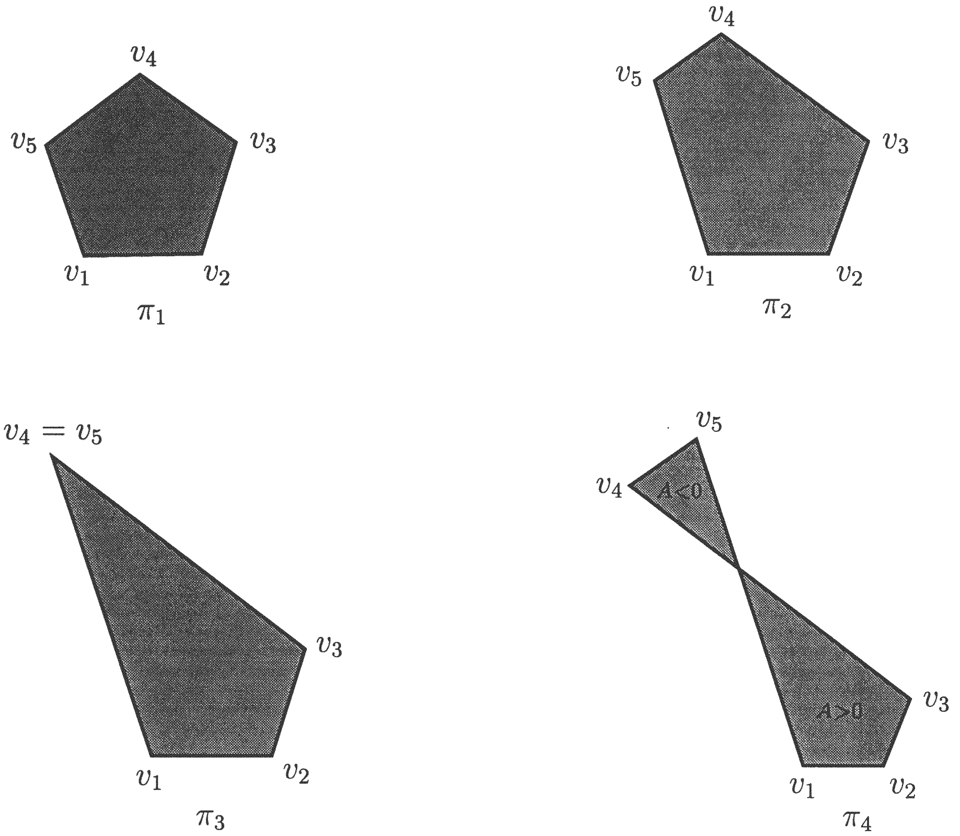

Figure 3. Polygons in $P_n^{\mathcal{L}}$: two in $\mathcal{C}_n^{\mathbb{L}}$, one on the boundary of $\overline{\mathcal{C}}_n^{\mathbb{L}}$ and one outside $\overline{\mathcal{C}}_n^{\mathbb{L}}$ but in the positive cone of A.

The choice of $\mathbb{L}$ that will be of interest in our current application corresponds to the oriented lines $\mathbb{L}_{\mathrm{eq}} = \left(L_1^{\mathrm{eq}}, \dots, L_n^{\mathrm{eq}}\right)$ coming from a fixed regular n-gon labeled in cyclic order, such as L_j^{eq} being the nonnegative multiples of the vector $\mathrm{e}^{2\pi i j/n} \in \mathbb{C} \cong \mathbb{R}^2$. Then Proposition 4.2 applies to $\overline{\mathcal{H}}_n^{\mathrm{eq}} = \mathbb{P}\overline{\mathcal{C}}_n^{\mathbb{L}_{\mathrm{eq}}}$ and tells us that $\overline{\mathcal{H}}_n^{\mathrm{eq}}$ is a hyperbolic polyhedron, compact if n is odd and closed but with $n/2$ limit points at infinity if n is even. It follows that $\overline{\mathcal{H}}_4^{\mathrm{eq}}$ has infinite volume and all the rest have finite volume.

The usefulness of $\mathbb{L}_{\mathrm{eq}}$ stems from the fact that the convex polygons $\mathcal{P}_n^{\mathrm{eq}}$ of Section 3 would all lie in $\mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$ if they were merely centered and then rotated to have labeled sides parallel to the lines of $\mathbb{L}_{\mathrm{eq}}$. In other words, if $\mathbb{E}^+(2) = S^1 \ltimes \mathbb{C}$ is the subgroup of rotations and translations in $\mathcal{A}$, then $\mathcal{P}_n^{\mathrm{eq}}/\mathbb{E}^+(2) \cong \mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$. The remaining part of the group $\mathcal{A}$ is $\mathbb{R}^*$, acting by scaling the polygons, which corresponds precisely to the projectivization of $\mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$. Thus, writing $\mathcal{H}_n^{\mathrm{eq}} = \mathbb{P}\mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$ for the interior of $\overline{\mathcal{H}}_n^{\mathrm{eq}}$, we have the following proposition:

PROPOSITION 4.3. $\mathcal{P}_n^{\mathrm{eq}}/\mathcal{A} \cong \mathcal{H}_n^{\mathrm{eq}}$.

To finish our computation of $\mathcal{M}_n$, we must understand the action of $\mathbb{D}_n$ on $\mathcal{H}_n^{\mathrm{eq}}$. Recall that the residual action of $\mathbb{D}_n$ on $D_\pm^n$ was by cyclically permuting or reversing the order of an n-tuple in $D_\pm^n$, which amounts to performing such operations on the labeling of a polygon in $\mathcal{P}_n^{\mathrm{eq}}$, or on the vector of side lengths of an element of $\mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$.

PROPOSITION 4.4. *$\mathbb{D}_n$ acts isometrically on $\mathcal{H}_n^{\mathrm{eq}}$ with exactly one fixed point.*

Proof. Let us determine how the side lengths of a polygon in $\mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$ are related to the n-tuple of vertices. A point $(v_1, \ldots, v_n) \in \mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$ has side lengths $(\lambda_1, \ldots, \lambda_n)$ if $v_{j+1} - v_j = \lambda_j\, \mathrm{e}^{2\pi ij/n}$ for all $1 \leqslant j \leqslant n$. In other words, the n-tuple $(v_1, \ldots, v_n)$ must be given by

$$(v_1, v_1 + \lambda_1\, \mathrm{e}^{2\pi i/n}, \cdots, v_1 + \lambda_1\, \mathrm{e}^{2\pi i/n} + \cdots + \lambda_{n-1}\, \mathrm{e}^{2\pi i(n-1)/n})$$

and the condition on the last side of the polygon tells us that

$$v_1 = v_1 + \lambda_1\, \mathrm{e}^{2\pi i/n} + \cdots + \lambda_n\, \mathrm{e}^{2\pi i}$$

or

$$\lambda_1\, \mathrm{e}^{2\pi i/n} + \cdots + \lambda_n = 0. \tag{4.1}$$

Note that the real and imaginary parts of this last equation give two constraints on the real numbers $(\lambda_1, \ldots, \lambda_n)$, so the solution set is indeed $(n-2)$-dimensional. Given a $(\lambda_1, \ldots, \lambda_n)$ satisfying (4.1), we can find the vertices of the corresponding polygon by the above formulæ once we know v_1, which is determined from the centering condition

$$0 = v_1 + \cdots + v_n = nv_1 + \sum_{j=1}^{n-1} (n-j)\lambda_j\, \mathrm{e}^{2\pi ij/n}.$$

In fact, if we add the constraint equation to this we get the more convenient form

$$v_1 = \frac{-1}{n} \sum_{j=1}^{n} (n-j+1)\lambda_j\, \mathrm{e}^{2\pi ij/n}.$$

Acting by the generator of the action of the $\mathbb{Z}_n \subset \mathbb{D}_n$, we get a new point $(w_1, \ldots, w_n) \in \mathcal{C}_n^{\mathbb{L}_{\mathrm{eq}}}$ with side lengths $(\mu_1, \ldots, \mu_n) = (\lambda_2, \ldots, \lambda_n, \lambda_1)$. These new side lengths satisfy (4.1) – just multiply (4.1) for the original side lengths by $\mathrm{e}^{-2\pi i/n}$ – and the new vertices will be built up from

$$w_1 = \frac{-1}{n} \sum_{j=1}^{n} (n-j+1)\mu_j\, \mathrm{e}^{2\pi ij/n} \quad \text{and} \quad w_{j+1} = w_j + \mu_j\, \mathrm{e}^{2\pi ij/n}. \tag{4.2}$$

Similarly, the generator of the action of the $\mathbb{Z}_2 \subset \mathbb{D}_n$ acts by changing the side lengths from $(\lambda_1, \ldots, \lambda_n)$ to $(\mu_1, \ldots, \mu_n) = (\lambda_{n-1}, \ldots, \lambda_1, \lambda_n)$; the constraint equation for these new side lengths remains valid as it is the complex conjugate of the original constraint. The new vertices will continue to be given by the formulæ (4.2). In particular, both of the generators and thus all of $\mathbb{D}_n$ act fixing the regular n-gon with $\lambda_1 = \cdots = \lambda_n$ – and this is the only polygon fixed by $\mathbb{D}_n$.

Furthermore, the two generators given above for $\mathbb{D}_n$ act on any polygon producing another which is congruent to or is a reflection of the original.

Thus the area is unchanged and, hence, $\mathbb{D}_n$ acts by isometries on $\mathcal{H}_n^{\text{eq}}$. This can also be seen with a direct (if tedious) computation from the definition of A, the action of the generators of $\mathbb{D}_n$ on the n-tuples of side lengths and the corresponding action on the vertices. □

COROLLARY 4.5. *The moduli space of boundary compactifications* $\mathcal{M}_n = \mathbb{D}_n \backslash \mathcal{H}_n^{\text{eq}}$ *is a contractible hyperbolic orbifold.*

Proof. It only remains to verify the assertion of contractibility. For this, observe that we can define a continuous, $\mathbb{D}_n$-equivariant retraction of $\mathcal{H}_n^{\text{eq}}$ to the fixed point by moving every point along the unique geodesic connecting it to the fixed point. This then induces a continuous retraction of $\mathbb{D}_n \backslash \mathcal{H}_n^{\text{eq}}$ to a point. □

We shall finish by giving some combinatorial information about the polyhedron $\overline{\mathcal{H}}_n^{\text{eq}}$ in general and computing the moduli space $\mathcal{M}_n$ explicitly in dimensions $n = 3$, 4, 5 and 6. As described above, $\overline{\mathcal{H}}_n^{\text{eq}}$ has n codimension one faces, each corresponding to one of the sides having zero length. It is in fact easier to index the faces of $\overline{\mathcal{H}}_n^{\text{eq}}$ of arbitrary codimension k by the $(n-k)$-tuples of sides which will *not* be degenerated. Following Bavard and Ghys, we call a cyclically ordered $(n-k)$-tuple of indices $(j_1, \ldots, j_{n-k})$ *compatible* if all of the ordered angles $\angle(L_{j_1}^{\text{eq}}, L_{j_2}^{\text{eq}}), \ldots, \angle(L_{j_{n-k}}^{\text{eq}}, L_{j_1}^{\text{eq}})$ are strictly less than π, which is the condition that allows us to simultaneously degenerate all sides with indices not in $\{j_1, \ldots, j_{n-k}\}$. Hence, faces of codimension k are in one-to-one correspondence with compatible $(n-k)$-tuples. Furthermore, a face of dimension k is contained in a face of dimension ℓ, where $k < \ell$, if and only if the corresponding compatible $(k+3)$-tuple is contained in the corresponding compatible $(\ell+3)$-tuple. (Recall that $\dim \mathcal{H}_n^{\text{eq}} = n - 3$.)

We can use a similar notation for the limit points at infinity of the polyhedron $\overline{\mathcal{H}}_n^{\text{eq}}$. These points are in one-to-one correspondence with pairs of parallel lines in $\mathbb{L}_{\text{eq}}$ and we shall index them by the pair (j_1, j_2) of indices of these parallel lines. Then, just as for the finite faces of $\overline{\mathcal{H}}_n^{\text{eq}}$, a sequence of polygons in $\overline{\mathcal{H}}_n^{\text{eq}}$ approaches a point with label (j_1, j_2) if and only if the lengths of all of the sides not numbered j_1 or j_2 are tending to zero. It follows that this point at infinity is a limit point of a face of $\overline{\mathcal{H}}_n^{\text{eq}}$ with corresponding compatible r-tuple $(j_1', \ldots, j_r')$ if and only if $\{j_1, j_2\} \subset \{j_1', \ldots, j_r'\}$.

For our choice of oriented lines $\mathbb{L}_{\text{eq}}$, an ordered set of indices will be compatible if and only if every successive pair differs, modulo n, by less than $n/2$. If, on the other hand, one successive pair of some ordered set of indices differs by an amount not less than $n/2$, then the entire set must lie in an interval of length at most $\lfloor n/2 \rfloor$, the greatest integer less than or equal to $n/2$. Thus, there are $n\binom{\lfloor n/2\rfloor}{r-1}$ incompatible r-tuples, or $\binom{n}{r} - n\binom{\lfloor n/2\rfloor}{r-1}$ compatible ones. (Note that this formula is relevant only if $r \geqslant 3$ and that, as usual, $\binom{n}{r} = 0$ if $r > n$.)

We can build up $\overline{\mathcal{H}}_n^{\text{eq}}$ inductively as follows. First, there will be $\binom{n}{3} - n\binom{\lfloor n/2\rfloor}{2}$ vertices and, if n is even, $n/2$ limit points at infinity – by abuse of notation, we will include these points at infinity in our construction of $\overline{\mathcal{H}}_n^{\text{eq}}$, as it simplifies the description and, in any case, the entire boundary of $\overline{\mathcal{H}}_n^{\text{eq}}$ must be removed to leave $\mathcal{H}_n^{\text{eq}}$. A given vertex will be connected to another vertex or to a point at infinity by a one-dimensional face of $\overline{\mathcal{H}}_n^{\text{eq}}$ if the union of their labels has four elements. An easy count shows that every vertex is connected by one-dimensional faces to exactly $n-3$ other vertices if n is odd and to $n-6$ vertices and 3 points at infinity if n is even. This is then the 1-skeleton of $\overline{\mathcal{H}}_n^{\text{eq}}$. The higher dimensional skeleta are constructed in a uniform manner: any subcomplex of the r-skeleton which is homeomorphic to an r-sphere is filled in with a $(r+1)$-ball in the $(r+1)$-skeleton if and only if the union of the $(r+3)$-tuples labeling its r-dimensional faces has $r+4$ elements.

Let us describe in detail the low-dimensional cases. When we provide pictures, we will only label the vertices and the points at infinity, as all other labels can be constructed from these.

$\boxed{n=3}$ Then $\mathcal{H}_3^{\text{eq}}$ is a point, the $\mathbb{D}_3$ action is trivial and $\mathcal{M}_3$ is also a point.

$\boxed{n=4}$ $\mathcal{H}_4^{\text{eq}}$ is an infinite line, where the two points at infinity correspond to the two pairs of parallel opposite sides in $\mathbb{L}_{\text{eq}}$. The generator of the $\mathbb{Z}_4 \subset \mathbb{D}_4$ acts interchanging these two points at infinity, while all of $\mathbb{D}_4$ fixes one point, representing the square. Thus $\mathcal{M}_4$ is a closed, half-infinite line.

$\boxed{n=5}$ $\overline{\mathcal{H}}_5^{\text{eq}}$ is of dimension two and compact. Its vertices correspond to the compatible triples $(1,2,4)$, $(1,3,4)$, $(1,3,5)$, $(2,3,5)$ and $(2,4,5)$, and each is connected to two others by a one-dimensional face. The $\mathbb{Z}_5 \subset \mathbb{D}_5$ action cyclically permutes these compatible triples, hence also the codimension one faces of $\overline{\mathcal{H}}_5^{\text{eq}}$, while one of the $\mathbb{Z}_2$'s in $\mathbb{D}_5$ acts by reflection fixing $(1,2,3,4)$ and $(2,3,5)$ but exchanging $(1,2,3,5)$ with $(2,3,4,5)$, $(1,2,4,5)$ with $(1,3,4,5)$, $(1,3,5)$ with $(2,4,5)$ and $(1,3,4)$ with $(1,2,4)$. Figure 4 shows $\overline{\mathcal{H}}_5^{\text{eq}}$ with vertices labeled. Also shown are the fixed point P_0 and ten fundamental domains for the $\mathbb{D}_5$-action.

$\boxed{n=6}$ $\overline{\mathcal{H}}_6^{\text{eq}}$ is 3-dimensional and has three limit points at infinity, corresponding to the pairs of indices $(1,4)$, $(2,5)$ and $(3,6)$. It also has two vertices at finite

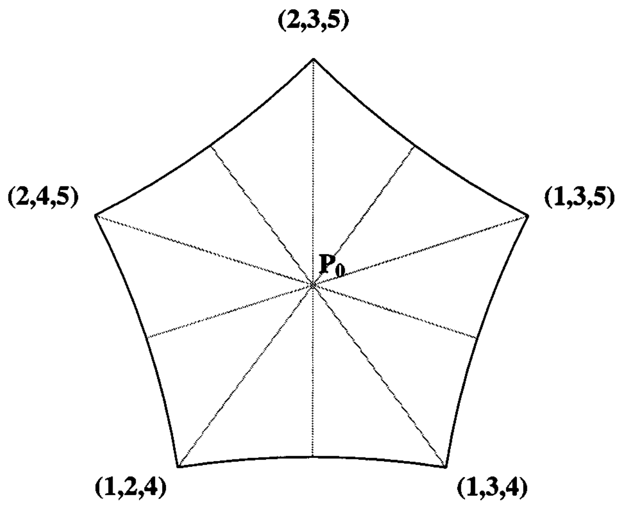

Figure 4. $\overline{\mathcal{H}}_5^{\text{eq}}$ with fundamental domains for the $\mathbb{D}_5$-action.

distance, corresponding to the compatible triples $(1,3,5)$ and $(2,4,6)$. There are nine faces of codimension two (edges), connecting the points at infinity in an infinite triangle and each of the finite vertices to all three points at infinity. Finally, there are six codimension one faces, corresponding to $(1,2,3,4,5),\ldots,(2,3,4,5,6)$. The $\mathbb{D}_6$ action can be computed from its effect on the faces, which are moved as described by the permutation action of $\mathbb{D}_6$ on the face labels. Figure 5 gives two views of $\overline{\mathcal{H}}_6^{\text{eq}}$, with vertices and points at infinity labeled.

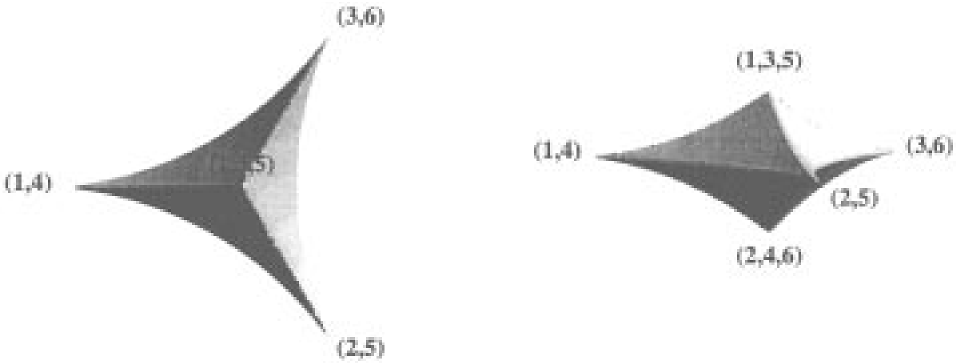

Figure 5. Top and side views of $\overline{\mathcal{H}}_6^{\text{eq}}$.

The fixed point P_0 lies at the midpoint of the straight line connecting the vertices $(1,3,5)$ and $(2,4,6)$. A fundamental domain for the action of $\mathbb{D}_6$ is formed, for example, by the union of all the radial straight lines connecting P_0 to that part of the face $(1,2,3,4,5)$ which is closer to $(1,4)$ than to $(2,5)$.

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