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Boundary compactifications of $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$

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Abstract. We construct a class of normal projective embeddings of $PSL(2, k)$, for $k = \mathbb{R}$ and $\mathbb{C}$, which we call *boundary compactifications* of $SL(2, k)$. These arise essentially as the Zariski closures of orbits in $(\mathbb{P}_k^1)^n$ under the diagonal action of $SL(2, k)$. In addition, we determine precisely when our examples can be $SL(2, k)$-homeomorphic, showing that the resulting deformation space is a countable union of positive-dimensional families.

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1. Introduction

The question that originally started this work was the following: given a connected semisimple Lie group G with finite center and no compact factors and a compact subgroup $K \subset G$, does there exist an equivariant embedding of G/K into a compact connected manifold M on which G acts by diffeomorphisms? Or, given that G is naturally an algebraic group with the Zariski topology induced by its adjoint representation, does there exist an equivariant embedding of G/K into an irreducible projective algebraic variety on which G acts algebraically? Moreover, if such an embedding exists, is it (essentially) unique?

Similar questions have been studied extensively for Kähler manifolds ([Hu-S]) and in the algebraic geometric setting over an algebraically closed field (see [A], [Br1], [Br2], [Br-L-V], [DC-Pr1], [DC-Pr2], [L-V], [MJ2] and [Pop], for example).

In particular, a very complete result was obtained in the early 1980's by Luna and Vust [L-V], who gave a classification of all normal embeddings of G/H, where G is a reductive algebraic group over an algebraically closed field with a factorial coordinate ring and H is an algebraic subgroup. In their terminology, a *normal embedding* of G/H is an irreducible normal algebraic variety with an action of G and which contains G/H as an open orbit. For the special case $G = SL(2, \mathbb{C})$ and

$H = \{e\}$, they gave an explicit classification of the embeddings in terms of diagrams which encode combinatorial data for the local rings of the orbits. Then in the late 1980's, L. Moser-Jauslin [MJ2] extended the combinatorial classification in terms of diagrams to the case in which H is any finite subgroup of $SL(2, \mathbb{C})$ and looked at the geometric properties that can be read from the diagrams.

In this paper, we give a geometric interpretation of some of the diagrams that appear in [MJ2], in particular of those which have nodes only at the ends (that is, the ones which correspond to embeddings which are complete and have a unique one-dimensional orbit). Even though we use algebraic methods to prove some of the properties of these embeddings (such as irreducibility and normality – see Theorem 3.4), our approach is very geometric in nature and gives results over both $\mathbb{R}$ and $\mathbb{C}$.

Definition 1.1. Let G be a connected k-algebraic group. An *algebraic compactification* X of G is a projective embedding of G/H, where H is a finite subgroup of G. (Thus in particular $\dim G = \dim X$.) If X is also a smooth algebraic variety then we say that X is a *smooth algebraic compactification*. Two algebraic compactifications X and Y of G are *topologically G-equivalent* if there exists a homeomorphism $\varphi : X \to Y$ which commutes with the G-actions.

In this paper we work exclusively with $G = SL(2, k)$ for $k = \mathbb{R}$ or $\mathbb{C}$ and we look at a particular kind of algebraic compactification. The idea behind this very simple construction is as follows. In the Furstenberg compactification of symmetric spaces [F] one considers an absolutely continuous measure on the boundary of a symmetric space. Here we go in completely the opposite direction and look instead at a measure which is as singular as possible, namely an atomic measure supported on a finite number of points. So let $\mathscr{X}$ be the boundary of the non-compact symmetric space with isometry group G, endowed with the usual G-action; as a k-algebraic variety, $\mathscr{X}_k$ is nothing other than $\mathbb{P}^1_k$ with the projectivized linear action. (Here and in the following the subscript k is used merely to emphasize, when necessary, the field with which we are working.) Letting G act on $\mathscr{X}^n$ diagonally, we can give the following

Definition 1.2. A *boundary compactification* X of G is the Zariski closure of the orbit of a point $p \in \mathscr{X}^n$ whose stabilizer in G is zero-dimensional. We say that X has *embedding dimension* n if X contains $n + 1$ G-orbits of positive codimension.

To relate the boundary compactifications we construct with the normal embeddings described by Luna, Vust and Moser-Jauslin, we prove the following

Theorem 3.4. *Any boundary compactification of embedding dimension $n > 3$ is an algebraic compactification of G with known, one-dimensional singular set.*

As mentioned above, we are also interested in to what extent algebraic compactifications are isomorphic. For our boundary compactifications, we shall prove that there is a non-empty Zariski open set $\mathscr{D}^{n-3} \subset k^{n-3}$ with an action of the symmetric group on n letters S_n such that

Theorem 4.4. *The set of topological G-isomorphism classes of boundary compactifications of embedding dimension n is in natural bijective correspondence with* $\mathscr{D}^{n-3}/S_n$.

We shall also relate boundary compactifications which arise as Zariski closures of orbits of points in $\mathscr{X}^n$ and $\mathscr{X}^m$ with $n \neq m$, see Theorem 4.5.

The above Theorem 4.4 essentially tells us that the boundary compactifications of G are far from unique as G-spaces – in fact they admit a positive-dimensional family $\mathscr{M}_n$ of deformations, even fixing the embedding dimension. We describe completely this moduli space $\mathscr{M}_n$ in companion papers [I-Por1] and [I-Por2], for the case $k = \mathbb{R}$.

Notice that even though our starting point was a finitely supported atomic measure, our statements are not in the measurable category, but really live in the topological one, as the distinguishing features of these compactifications depend on a certain continuous behavior as one approaches the boundary.

One natural question one might ask next is whether the boundary compactifications we construct might be non-equivalent (modulo the S_n-action) also as spaces with an action of some subgroup of G, and, in this case, for which subgroups might this picture arise. One natural candidate is a lattice $\Gamma \subset G$; clearly if we were looking only at algebraic isomorphisms then, by the Borel Density Theorem, algebraic G-isomorphism classes would coincide with Γ-isomorphism classes. On the other hand, a close observation of the proof of the topological non-equivalence of our boundary compactifications shows that the result depends only on the existence of enough hyperbolic fixed points in $\mathscr{X}$. A lattice has a dense set of hyperbolic fixed points in $\mathscr{X}$ and this is enough for part of Theorem 4.1, but a much stronger notion of density is necessary for the remainder, and we are currently investigating this issue.

The very elementary ingredients used in our construction of boundary compactifications of G are the cross ratio and the action of G on the boundary of 2- and 3-dimensional real hyperbolic space. There is a notion of cross ratio due to Korányi and others ([Ko-Re] and [C-D-Ko-Ri]), which should suffice at least for other $\mathbb{R}$-rank one groups. One problem still to overcome is the lack of three-point transitivity for the action of the $\mathbb{R}$-rank one groups over $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ on the boundaries of their associated symmetric spaces. While this poses new difficulties, it also may introduce enough complexity in the actions for some of the spaces to be distinguishable purely measurably.

The remainder of this paper is organized as follows. § 2 gives some general basic facts and analyzes the lowest dimensional example of a boundary compactification which is qualitatively somewhat different from the higher dimensional cases. These other examples are examined in § 3, while § 4 studies the deformations of actions possible in our families of examples, by deriving a basic result about when two boundary compactifications can be topologically G-isomorphic.

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2. The lowest dimensional case

We start by summarizing some very well known elementary facts about SL_2-actions. Most of what we are going to need can be found in much greater generality in any standard reference for hyperbolic geometry such as, for instance, [Be] or [Ka].

Recall that we are working with the group $G = SL_2$, which we shall write $G_k = SL(2,k)$ when we wish to emphasize the field k. When $k = \mathbb{R}$ or $\mathbb{C}$, $PG_k = G_k/\pm Id$ is the orientation-preserving isometry group of the hyperbolic space $\mathscr{H}^2_{\mathbb{R}}$ or $\mathscr{H}^3_{\mathbb{R}}$, respectively; we shall write $\mathscr{H}$ when we wish to work with whichever hyperbolic space is of the appropriate dimension for the current choice of k. It will be necessary below to work also with the full group $GM(\mathbb{R})$ of isometries of $\mathscr{H}^2_{\mathbb{R}}$, which includes even the isometries that reverse orientation; it will be convenient to use the notation $\tilde{G}_k = GM(\mathbb{R})$ if $k = \mathbb{R}$ and $\tilde{G}_k = G_{\mathbb{C}}$ if $k = \mathbb{C}$, with the agreement that the dependence on the particular field k will be omitted when not relevant. We shall use as well the subgroup

$$A_k = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in k \setminus \{0\} \right\} \subset G_k$$

of dilations and the parabolic subgroup

$$P_k = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in k \setminus \{0\}, b \in k \right\} \subset G_k \,.$$

We denote by $\mathscr{X}$ the boundary of the hyperbolic space $\mathscr{H}$, and note that $\mathscr{X}_k = \mathbb{P}^1_k$ with the (algebraic) projectivized linear action of G_k.

Much of our geometric work rests on the fact that no non-trivial element of G fixes more than two points in $\mathscr{X}$; the elements g which do fix two points are called *hyperbolic* or *loxodromic* and they also fix setwise a geodesic in $\mathscr{H}$. The fixed points of g on $\mathscr{X}$ are the endpoints of this geodesic and all other points are moved away from one of the fixed points – the *repelling fixed point* – and towards the other – the *attracting fixed point*. It is important that for any $a, b \in \mathscr{X}$, $a \neq b$, there is a one-parameter subgroup of G which has a as the repelling fixed point and b as the attracting.

Now let G act on $\mathscr{X}^n$ diagonally and define the following G-invariant sets, which we shall use extensively in the sequel:

(1) $\Delta^n_0 = \{(z, z, \dots, z) \in \mathscr{X}^n\} \simeq \mathscr{X}$;

(2) $\Delta_j^n = \{(z_1, z_2, \ldots, z_n) \in \mathscr{X}^n : z_l = z_m$ whenever $l, m \neq j$, while $z_i \neq z_j$ for $i \neq j\}$ $\simeq \mathscr{X}^2 \backslash \Delta_0^2$, e.g., $\Delta_1^n = \{(z, w, \ldots, w) \in \mathscr{X}^n : z \neq w\}$;

(3) $\Delta_{ij}^n = \{(z_1, z_2, \ldots, z_n) \in \mathscr{X}^n : z_l = z_m$ whenever $\{l, m\} \neq \{i, j\}$, while $z_j \neq z_l \neq z_i$ $\neq z_j$ whenever $l \neq i, j\}$, e.g., $\Delta_{12}^n = \{(z_1, z_2, w, \ldots, w) \in \mathscr{X}^n : z_1 \neq z_2 \neq w \neq z_1\}$;

(4) $D^n = \{(z_1, z_2, \ldots, z_n) \in \mathscr{X}^n : z_i \neq z_j$ if $i \neq j\}$.

(Here $\simeq$ means isomorphism as algebraic – and hence topological – G-spaces.)

The action on $\mathscr{X}^3$ is particularly easy and is described in the next Proposition.

Proposition 2.1. *When $k = \mathbb{R}$ or $\mathbb{C}$,*

(a) *$\tilde{G}$ is transitive on D^3 and PG acts freely there.*

(b) *There is a decomposition of $\mathscr{X}^3 \backslash D^3$ into G-orbits as follows:*

$$\mathscr{X}^3 \backslash D^3 = \Delta_1^3 \cup \Delta_2^3 \cup \Delta_3^3 \cup \Delta_0^3 .$$

(c) *$\Delta_j^3 \simeq G/A$ when $j = 1, 2$ or 3 and $\Delta_0^3 \simeq G/P$.*

And when $k = \mathbb{R}$ only

(d) *$D_{\mathbb{R}}^3$ decomposes further under the action of $G_{\mathbb{R}}$ into the two orbits D_+^3 and D_-^3, corresponding to different orientations.*

Proof. Since there are no elements in G other that $\pm Id$ which fix three points in $\mathscr{X}$, the action of PG on $\mathscr{X}^3$ is free; as PG is connected and acts smoothly, the orbit of a point in D^3 must be a component. But $D_{\mathbb{R}}^3$ has two components, differing by the cyclic order of the triples on $\mathscr{X}_{\mathbb{R}}$, and $D_{\mathbb{C}}^3$ only one. Further, $\tilde{G}_{\mathbb{R}} = GM(\mathbb{R})$ consists of $PG_{\mathbb{R}}$ and a coset by a reflection, which reverses the cyclic order of triples on $\mathscr{X}_{\mathbb{R}}$.

Now $\Delta_0^3 \simeq \mathscr{X} \simeq G/P$ as G-spaces. Moreover, A consists of hyperbolic elements which fix two elements $z_0, w_0 \in \mathscr{X}$, with $z_0 \neq w_0$. Thus $\Delta_3^3 = \{(z, z, w) \in \mathscr{X}^3 : z \neq w\}$ $= G \cdot (z_0, z_0, w_0) \simeq G/A$, and analogously for $j = 1, 2$. □

Corollary 2.2. *$\mathscr{X}^3$ is a smooth algebraic compactification of SL_2.*

Proof. Certainly $\mathscr{X}^3$ is a smooth (hence normal), three-dimensional, irreducible algebraic variety, and we have just seen that it has an orbit with stabilizer $\mathbb{Z}_2$. □

3. The general case

Recall [Be] that given any four distinct complex numbers, $z_1, z_2, z_3, z_4 \in \mathbb{C}$ their cross ratio is defined by

$$c(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)},$$

and that c may even by defined in the obvious way when one of its arguments is ∞; hence c is in fact a rational function on $D^4_{\mathbb{C}}$. When restricted to the real points, c is also a real-valued rational function defined on $D^4_{\mathbb{R}}$. Furthermore, c is invariant under the action of $\tilde{G}$; conversely, any two points in D^4 which have the same cross ratio are in the same $\tilde{G}$-orbit. For future reference we state here a useful version of this fact, the proof of which is quite easy and can be found for example in [Be].

Lemma 3.1. *Let* $(z_1, z_2, w, z_3) \in D^4$, *with* $c(z_1, z_2, w, z_3) = t$. *If* $g \in \tilde{G}$ *is the element (uniquely determined, up to* $\pm Id$*) which sends* (z_1, z_2, z_3) *to* $(0, 1, \infty)$, *then* $gw = t$. *In other words, there exists* $g \in \tilde{G}$ *such that* $g(z_1, z_2, w, z_3) = (0, 1, t, \infty)$ *if and only if* $c(z_1, z_2, w, z_3) = t$.

It follows that the cross ratio is a surjective map from D^4 to $k \setminus \{0, 1\}$. Hence, for any $t \in k \setminus \{0, 1\}$ the set $c^{-1}(t)$ consists of one orbit of $\tilde{G}$ and two $G_{\mathbb{R}}$-orbits when $k = \mathbb{R}$, each corresponding to different orientations of the 4-tuple.

The first thing we must do to move to higher dimensions is to define a map which will be just a generalization of the classical notion of cross ratio. Let $c_n : D^n \to k^{n-3}$ be given by

$$c_n(z_1, z_2, w_1, \ldots, w_{n-3}, z_3) = (c(z_1, z_2, w_1, z_3), \ldots, c(z_1, z_2, w_{n-3}, z_3)) .$$

Let $\mathscr{D}^{n-3} = \{(t_1, t_2, \ldots, t_{n-3}) \in (k \setminus \{0, 1\})^{n-3} : t_i \neq t_j, \text{ for } 1 \le i, j \le n-3\}$; by Lemma 3.1, $c_n(D^n) \subset \mathscr{D}^{n-3}$. We collect some elementary facts about this generalized cross ratio.

Lemma 3.2. *For every* $t \in \mathscr{D}^{n-3}$, *the set* $D^n_t = c_n^{-1}(t) \subset D^n$ *is one* $\tilde{G}$*-orbit but two* $G_{\mathbb{R}}$*-orbits when* $k = \mathbb{R}$; *in both cases,* PG_k *acts freely. Also, the map* $c_n : D^n \to \mathscr{D}^{n-3}$ *is surjective.*

Proof. Fix $t = (t_1, \ldots, t_{n-3}) \in \mathscr{D}^{n-3}$ and observe that the sets

$$A_j = \{(z_1, z_2, w_1, \ldots, w_{n-3}, z_3) \in D^n : c(z_1, z_2, w_j, z_3) = t_j\}$$

are $\tilde{G}$-invariant, hence so also is

$$c_n^{-1}(t) = \bigcap_{j=1}^{n-3} A_j .$$

Conversely, if $p = (z_1, z_2, w_1, \ldots, w_{n-3}, z_3) \in c_n^{-1}(t)$, where $t = (t_1, \ldots, t_{n-3}) \in \mathscr{D}^{n-3}$, then we have $c(z_1, z_2, w_j, z_3) = t_j$ for $j = 1, 2, \ldots, n-3$. If $g \in \tilde{G}$ is such that $gz_1 = 0$, $gz_2 = 1$ and $gz_3 = \infty$, then $gw_j = t_j$ by Lemma 3.1, which implies that $(z_1, z_2, w_1, \ldots, w_{n-3}, z_3)$ is in the same $\tilde{G}$-orbit as $(0, 1, t_1, \ldots, t_{n-3}, \infty)$.

On the other hand, since $G_{\mathbb{R}}$ consists only of orientation-preserving Möbius transformations, D^n_t consists of two $G_{\mathbb{R}}$-orbits,

$$D^n_{t,+} = G_{\mathbb{R}} \cdot (0, 1, t_1, \ldots, t_{n-3}, \infty)$$

and

$$D^n_{t,-} = G_{\mathbb{R}} \cdot (0, -1, -t_1, \ldots, -t_{n-3}, \infty)$$

(corresponding to the opposite orientation).

The surjectivity of c_n is an immediate consequence of the prior statements. □

We want to examine now the behavior of the orbit closures of points in D^n.

Proposition 3.3. *Let $p \in D^n$ have $c_n(p) = t$. Then the closure in the Hausdorff topology of its $\tilde{G}$-orbit D^n_t is the set*

$$X^n_t = D^n_t \cup \left(\bigcup_{j=1}^{n} \Delta^n_j \right) \cup \Delta^n_0 .$$

If $k = \mathbb{R}$, then the two G-orbits $D^n_{t,\pm}$ have closures

$$X^n_{t,\varepsilon} = D^n_{t,\varepsilon} \cup \left(\bigcup_{j=1}^{n} \Delta^n_j \right) \cup \Delta^n_0 ,$$

where $\varepsilon = \pm$.

Proof. We shall see below in Theorem 3.4 that X^n_t is Zariski closed, so it is also closed in the Hausdorff topology. Thus $\overline{D^n_t} \subseteq X^n_t$. If $k = \mathbb{R}$, let $U_\pm$ be the connected components of D^n containing $D^n_{t,\pm}$, both open in $\mathscr{X}^n$. Then $X^n_{t,\varepsilon} = X^n_t \cap (\mathscr{X}^n \setminus U_{-\varepsilon})$ is closed, so also the closures of $D^n_{t,\pm}$ are contained in the respective sets $X^n_{t,\pm}$.

For the opposite inclusions, let $q = (z_1, \ldots, z_n)$ be in X^n_t or $X^n_{t,\pm}$ and write $\phi_{a,b}(\tau)$ for the one-parameter subgroup of G mentioned in § 2 with repelling fixed point $a \in \mathscr{X}$ and attracting fixed point $b \in \mathscr{X}$. For any $a \in \mathscr{X} \setminus \{z_1, \ldots, z_n\}$ and $b \in \mathscr{X}$,

$$\lim_{\tau \to \infty} \phi_{a,b}(\tau) \cdot q = (b, \ldots, b) \in \Delta_0 .$$

Also, for any $1 \leq j, k \leq n$ with $j \neq k$,

$$\lim_{\tau \to \infty} \phi_{z_j, z_k}(\tau) \cdot q = (z_k, \ldots, z_k, z_j, z_k, \ldots, z_k) \in \Delta_j$$

where the z_j is in the jth place. But the closure of a G-orbit is at least G-invariant, and the sets $\Delta_0, \Delta_1, \ldots, \Delta_n$ are G-orbits, so $(\bigcup_{j=1}^n \Delta^n_j) \cup \Delta^n_0$ is contained in the closures of X^n_t and either $X^n_{t,\pm}$. Thus $X^n_t \subseteq \overline{D^n_t}$ and $X^n_{t,\varepsilon} \subseteq \overline{D^n_{t,\varepsilon}}$, so the orbit closures are as claimed. □

From what we have observed so far, the actions of $\tilde{G}$ on D_t^n and $G_{\mathbb{R}}$ on $D_{t,\varepsilon}^n$, where $\varepsilon = \pm$, are transitive and have finite stabilizer. While these orbits (either the D_t^n's or the $D_{t,\varepsilon}^n$'s) form a foliation of D^n, their topological closures all intersect, as they all have the same boundary. In other words,

$$\bigcap_{t \in \mathscr{D}^{n-3}} X_t^n = \bigcap_{t \in \mathscr{D}^{n-3}} X_{t,\varepsilon}^n = \left(\bigcup_{j=1}^{n} \Delta_j^n \right) \cup \Delta_0^n .$$

This groundwork now enables us to realize the examples we were seeking.

Theorem 3.4. *For all $t \in \mathscr{D}^{n-3}$, X_t^n is an algebraic compactification of G whose singular set is Δ_0^n.*

Proof. Fix such a $t \in \mathscr{D}^{n-3}$. What remains is to show that X_t^n is an irreducible normal projective k-algebraic variety with the specified singular set. We shall do this by exhibiting X_t^n as the zero set of an ideal generated by cross ratios with their denominators cleared. In fact it suffices to do this on each element in an affine open cover of $\mathscr{X}^n$; we shall thus work in the coordinate path $U = k^n \subset (k \cup \infty)^n = \mathscr{X}^n$, the other patches being similar, and proceed step by step.

Step 1: Zariski closed. Choosing any $p = (z_1, \ldots, z_n) \in D_t^n$, define constants $s_{ijuv} = c(z_i, z_j, z_u, z_v)$. Consider the ideal I generated by the homogeneous polynomials

$$\xi_{ijuv}(x_1, \ldots, x_n) = (x_i - x_u)(x_j - x_v) - s_{ijuv}(x_i - x_j)(x_u - x_v)$$

where $1 \leq i, j, u, v \leq n$ are distinct. These polynomials vanish on the $\tilde{G}$-orbit of p, as they do trivially on $U \cap (\Delta_0^n \cup \bigcup_{j=1}^n \Delta_j^n)$, so $X_t^n \cap U$ is contained in the zero set V of I.

For the converse, note that for points in D^n we can turn the equations $\xi_{ijuv} = 0$ into statements about cross ratios, so $X_t^n \cap U \cap D^n = V \cap D^n$. Next, take a point $q = (x_1, \ldots, x_n) \in V \setminus D^n$. If there are three or more distinct values taken by the coordinates of q, then choose indices i, j, u and v in such a way that $x_i \neq x_j \neq x_u = x_v \neq x_i$. Then

$$0 = \xi_{ijuv}(x_1, \ldots, x_n) = (x_i - x_u)(x_j - x_v) - s_{ijuv}(x_i - x_j)(x_u - x_v) \neq 0$$

is a contradiction. Similarly, $x_i = x_j \neq x_u = x_v$ would also contradict the same equation. It follows that the only points in $V \setminus D^n$ must be in some Δ_j^n. Hence V is exactly equal to $X_t^n \cap U$ and so the latter is Zariski closed in U.

Step 2: The smooth part. We write V_{sm} for the smooth part of V. For this step of the proof let us write $\xi_j = \xi_{12j(j+1)}$ and $s_j = s_{12j(j+1)}$. Then at a point $q = (x_1, \ldots, x_n) \in V$ we compute the gradient

$$\nabla \xi_j = ((x_2 - x_{j+1}) - s_j(x_j - x_{j+1}))\frac{\partial}{\partial x_1} + ((x_1 - x_j) + s_j(x_j - x_{j+1}))\frac{\partial}{\partial x_2}$$
$$+ ((x_{j+1} - x_2) - s_j(x_1 - x_2))\frac{\partial}{\partial x_j} + ((x_j - x_1) + s_j(x_1 - x_2))\frac{\partial}{\partial x_{j+1}}$$

and note that if $q \in \Delta_1^n$ the coefficient of $\frac{\partial}{\partial x_j}$ is non-zero. In fact, if $q \in D_t^n$ then using $\xi_j = 0$ we find that

$$(x_{j+1} - x_2) - s_j(x_1 - x_2) = (x_{j+1} - x_2) - \frac{(x_1 - x_j)(x_2 - x_{j+1})}{(x_1 - x_2)(x_j - x_{j+1})}(x_1 - x_2)$$
$$= \frac{(x_{j+1} - x_2)(x_1 - x_{j+1})}{(x_j - x_{j+1})} \neq 0$$

so again the coefficient of $\frac{\partial}{\partial x_j}$ is non-zero. Letting j range from 3 to $n-1$, we must have at least $n-3$ linearly independent gradients of elements of I along $U \cap (D_t^n \cup \Delta_1^n)$ and in fact along all of $V \setminus \Delta_0^n$, as there was nothing particularly special about Δ_1^n here. But since V_{sm} is open in V and of algebraic dimension at least three – D_t^n is a smooth manifold of dimension three over k – we must have $V \setminus \Delta_0^n \subset V_{sm}$. Since the gradients of all of the generators of I are identically zero on Δ_0^n, we find in fact that $V \setminus \Delta_0^n = V_{sm}$.

Step 3: Irreducibility. Note that V_{sm} is connected in the Hausdorff topology and so is irreducible quasi-projective. But V_{sm} contains the set $\Delta_1^n \cap U$ so its Zariski closure must contain $\overline{\Delta_1^n \cap U}^Z = (\Delta_1^n \cup \Delta_0^n) \cap U$. Thus V is irreducible.

Step 4: Normality. To verify the normality of a variety whose singular set has codimension two, we need to check the "condition S_2 of Serre", see [Ha]. Let us first exhibit a particular regular sequence of length two for the ring $\mathcal{O}(V) = k[x_1, \ldots, x_n]/\sqrt{I}$ of regular functions itself. Set $\alpha = x_1 - x_2$ and note that its zero set H_α is a hyperplane which intersects V in $H_\alpha \cap V = (\bigcup_{j=3}^n \Delta_j \cup \Delta_0) \cap U$. Since $\mathcal{O}(V)$ is a domain, α is certainly not a zero divisor.

For $j > 1$, $\{x_1 - x_i \mid 1 < i \leq n, i \neq j\}$ is a set of generators for the ideal of $\Delta_j \cup \Delta_0$, so the ideal $\sqrt{\sqrt{I} + (\alpha)}$ of $H_\alpha \cap V$ is generated by $x_1 - x_2$ and

$$\{(x_1 - x_i)(x_1 - x_j) \mid 1 < i < j \leq n\}.$$

But for any such $1 < i < j \leq n$

$$(x_1 - x_i)(x_1 - x_j) = (x_1 - x_i)(x_1 - x_2) + (x_1 - x_i)(x_2 - x_j)$$
$$= (x_1 - x_i)\alpha + \xi_{12ij} + s_{12ij}(x_1 - x_2)(x_i - x_j) \in I + (\alpha).$$

Hence $\sqrt{\sqrt{I}+(\alpha)} \subseteq I+(\alpha) \subseteq \sqrt{I}+(\alpha) \subseteq \sqrt{\sqrt{I}+(\alpha)}$ and it follows that $\sqrt{I}+(\alpha)$ is a radical ideal.

Next, pick numbers $a_1, \ldots, a_n \in k \setminus 0$ satisfying $\sum_{j=1}^{n} a_j = 0$ and define $\beta = \sum_{j=1}^{n} a_j x_j$; note that the zero set H_β of β is a hyperplane satisfying $H_\alpha \cap H_\beta \cap V = \Delta_0 \cap U$. But this implies that β is not a zero divisor in $\mathcal{O}(V)/(\alpha)$, since it would then have to vanish on a component of $H_\alpha \cap V$, which we have seen is not the case.

As the singular set of V is $\Delta_0 \cap U$, we can conclude by noting that α, β remains the required regular sequence of length two when we localize at the principal ideal of $\Delta_0 \cap U$ or of any point thereon. □

We can hence refine the definition of boundary compactification given in the introduction as follows.

Definition 3.5. A *boundary compactification* of G_k is either $\mathscr{X}_k^3$ or one of the algebraic compactifications of the form X_t^n for some $n > 3$ and $t \in \mathscr{D}^{n-3}$; n, or 3 in the case of $\mathscr{X}^3$, will be called the *embedding dimension* of the compactification.

This terminology is motivated by the essential use in our construction of the boundary of the symmetric space associated to G_k.

Remark. In view of Theorem 3.4 we conclude that our boundary compactifications give a geometric interpretation of some of the embeddings described in [MJ1] and [MJ2]. In particular, it was pointed out to us by Vust and Moser-Jauslin that our boundary compactifications correspond to their combinatorial diagrams with nodes only at the ends.

4. Deformations of actions

This section will be concerned with determining when two boundary compactifications can be isomorphic, with respect to various notions of isomorphism. The following is the basic result in this investigation.

Theorem 4.1. *Let $p, p' \in D^n$ and $q \in D^m$, for $m \neq n$. Then*

(a) *$\overline{G \cdot p}$ and $\overline{G \cdot q}$ are never isomorphic as topological (and hence algebraic) G-spaces.*

(b) *A map $\varphi : \overline{G \cdot p} \to \overline{G \cdot p'}$ is a topological G-isomorphism if and only if it is a permutation $\sigma \in S_n$.*

Proof. (1) Our decomposition of $\overline{G \cdot p}$ above indicates that it contains exactly $n+3$ G_k-orbits if $K = \mathbb{R}$ and $n+2$ orbits if $K = \mathbb{C}$. Hence boundary compactifications with different embedding dimensions cannot be G-isomorphic.

(2) Clearly any permutation $\sigma \in S_n$ is a homeomorphism commuting with the G-action on $\mathscr{X}^n$ and hence gives an isomorphism of G-spaces.

To see the converse, start with a topological G-isomorphism $\varphi : \overline{G \cdot p} \to \overline{G \cdot p'}$. Write $p = (z_1, z_2, \dots, z_n)$ and let $g \in SL_2$ be a hyperbolic element fixing z_1 and z_2 for which z_1 is a repelling fixed point. Then

$$\varphi(z_1, z_2, \dots, z_2) = \lim_{i \to \infty} \varphi(g^i(z_1, z_2, \dots, z_n)) = \lim_{i \to \infty} g^i \varphi(z_1, z_2, \dots, z_n)$$
$$= \lim_{i \to \infty} g^i(z'_1, z'_2, \dots, z'_n) .$$

Now $(z_1, z_2, \dots, z_2) \in \Delta^n_1$, and $\varphi(\Delta^n_1) = \Delta^n_j$ for some j, since the Δ^n_j are the only orbits in any $\overline{G \cdot p'}$ with 1-dimensional stabilizer. But if none of the $z'_1, z'_2, \dots, z'_n$ equals z_1, then

$$\lim_{i \to \infty} g^i(z'_1, z'_2, \dots, z'_n) = (z_2, z_2, \dots, z_2) \in \Delta^n_0 .$$

Hence one of the coordinates of $(z'_1, z'_2, \dots, z'_n)$ has to be equal to z_1.

Repeating the same argument above with hyperbolic elements g which have $z_2, \dots, z_n$ in turn as a repelling fixed point, we gather that the only case in which we can possibly have the homeomorphism φ commute with the G-action is when $\{z_1, z_2, \dots, z_n\} = \{z'_1, z'_2, \dots, z'_n\}$, so that $\varphi(z_1, z_2, \dots, z_n) = (z'_1, z'_2, \dots, z'_n) = \sigma(z_1, z_2, \dots, z_n)$ for some $\sigma \in S_n$. □

While for $k = \mathbb{C}$ the above result is what we were looking for, when $k = \mathbb{R}$, it implies only that a G-equivariant homeomorphism $\varphi : X^n_t \to X^n_{t'}$, restricted to either of the two topological orbit closures $X^n_{t,\varepsilon}$ is a permutation σ_ε. *A priori* it might happen that $\sigma_+ \neq \sigma_-$, but the next Lemma shows that this is in fact impossible.

Lemma 4.2. *Let $k = \mathbb{R}$. If $\eta = \pm$ is fixed and $\sigma_\varepsilon : X^n_{t,\varepsilon} \to X^n_{t',\eta\varepsilon}$ for $\varepsilon = \pm$ are two permutations $\sigma_\varepsilon \in S_n$ which agree on the overlap $X^n_{t,+} \cap X^n_{t,-}$, then $\sigma_+ = \sigma_-$.*

Proof. By our decomposition of orbit closures, $X^n_{t,+}$ and $X^n_{t,-}$ intersect along their boundaries $(\bigcup^n_{j=i} \Delta^n_j) \cup \Delta^n_0$. Composing both permutations with σ_-^{-1}, we may as well study the permutation $\sigma' = \sigma_-^{-1}\sigma_+$ for which we have that

$$\sigma'|_{(\bigcup^n_{j=i} \Delta^n_j) \cup \Delta^n_0} = (\sigma_-^{-1}\sigma_+)|_{(\bigcup^n_{j=i} \Delta^n_j) \cup \Delta^n_0} = Id .$$

But saying that $\sigma'|_{\Delta^n_j} = Id$ is equivalent to saying that σ' is in the isomorphic copy $S_n(j)$ of S_{n-1} contained in S_n which fixes the j-th position. Since $\bigcap^n_{j=1} S_n(j) = Id$, we have that $\sigma' = Id$, so $\sigma_+ = \sigma_-$ as we wanted. □

Corollary 4.3. *Let $t, t' \in \mathscr{D}^{n-3}$ and $s \in \mathscr{D}^{m-3}$, for $m \neq n$. Then*

(a) *X^n_t and X^m_s are never isomorphic as topological G-spaces.*

(b) *A map $\varphi : X^n_t \to X^n_{t'}$ is a topological G-isomorphism if and only if it is a permutation $\sigma \in S_n$.*

In light of this result, it is paramount to define an action of S_n on $\mathcal{D}^{n-3}$. Let $(t_1, \ldots, t_{n-3}) \in \mathcal{D}^{n-3}$ and $\sigma \in S_n$; then we set

$$\sigma(t_1, \ldots, t_{n-3}) = c_n(\sigma(0, 1, t_1, \ldots, t_{n-3}, \infty)) .$$

This enables us to summarize our results so far in the following way:

Theorem 4.4. *The set of topological G-isomorphism classes of boundary compactifications with embedding dimension n is in natural bijective correspondence with* $\mathcal{D}^{n-3}/S_n$.

To conclude, let us note that the hyperplanes we remove from k^{n-3} to make $\mathcal{D}^{n-3}$ also represent boundary compactifications, just not of embedding dimension n. In fact, the Zariski closures of most G-orbits in $\mathcal{X}^n$ are boundary compactifications of embedding dimensions less than or equal to n. More precisely:

Theorem 4.5. (a) *For* $p \in \mathcal{X}^n$, *let* $p' \in D^l$ *be p with repeated coordinates removed. If* $l \geq 3$, *then* $\overline{G \cdot p}^Z$ *is isomorphic to a boundary compactification of embedding dimension l.*

(b) *For* $s, t \in k^{n-3}$, *let* $s' \in \mathcal{D}^l$ *and* $t' \in \mathcal{D}^m$ *be s and t with coordinates that are either repeated or equal to* 0 *or* 1 *removed. Let* $X^l_{s'} = \overline{c_l^{-1}(s')}$ *and* $X^m_{t'} = \overline{c_m^{-1}(t')}$. *Then* $X^l_{s'}$ *and* $X^m_{t'}$ *are isomorphic boundary compactifications if and only if* $l = m$ *and* s' *and* t' *are related by the action of an element of* S_l *on* $\mathcal{D}^l$ – *in which case* $X^l_{s'} = X^m_{t'}$ *has embedding dimension* $l + 3 = m + 3$.

Proof. This follows easily from what we have done above and the rather trivial facts that removing a few coordinates commutes with the diagonal G-action and that a subset defined by a certain number of coordinates being equal is Zariski closed. □

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