

CSUP MATH 307 SPRING 2011

TEST I REVIEW SHEET

1. DEFINITIONS

For each term, the meaning of the boxes is:

Type: What kind of object it is – a real number, a set, a matrix, *etc.*

Context: The environment in which this definition makes sense.

Def: A complete, formal definition of the object – make sure all variables in this definition are either the thing being defined, part of the *Context*, or have associated **quantifiers**.

Idea: A description in your own words of what the *essential idea* of this term is.

E.g.: A complete, careful, explicit example (or more than one) of this term.

C.f.: Names or descriptions of one or more theorems, constructions, algorithms, or other definitions in which this term is used.

The following is an example:

<i>[Context]</i> For $n \in \mathbb{N}$	a vector subspace of $\mathbb{R}^n$	<i>[Type]</i> is a subset $V \subseteq \mathbb{R}^n$
<i>[Def]</i> satisfying $\alpha \vec{v} + \beta \vec{w} \in V \quad \underline{\forall \vec{v}, \vec{w} \in V, \forall \alpha, \beta \in \mathbb{R}} .$		
<i>[Idea]</i> A <i>subspace</i> is a subset closed under linear combinations; <i>i.e.</i> , where we can manipulate vectors with the usual vector addition and scalar multiplication without worrying that the result may leave the subspace.		
<i>[e.g.]</i> The x -axis (or any line through the origin) is a subspace of $\mathbb{R}^2$; any line or plane through the origin is a subspace of $\mathbb{R}^3$		
<i>[c.f.]</i> The following are all vector subspaces of some appropriate $\mathbb{R}^n$: the null space, column space, and row space of a matrix; the kernel and range of a linear transformation; an eigenspace of a matrix.		

At the end of this document are several pages of these blank *definition forms* (with different size boxes, as may be necessary for terms with more or less elaborate definitions, or examples requiring more or less space, *etc.*). Feel free to copy and use them, or to make your own by hand or in your favorite word processor.

Here are terms for which this level of definition is appropriate (here “mat.” stands for “matrix”):

vector	span	eigenvalue	row space	linearly [in]dependent
matrix	rank	eigenvector	column space	linear transformation
inverse	basis	eigenspace	null space	characteristic polynom.
nullity	kernel	Markov chain	stochastic mat.	[in]homogeneous system
[di]graph	[R]REF	transpose	adjacency mat.	standard mat. (of a LT)
dimension	rotation	geometric mult.	transition mat.	steady state vector
determinant	diagonal	algebraic mult.	elementary mat.	linear combination
triangular	similar	diagonalizable	projection	dot product

2. THEOREMS/RESULTS/TECHNIQUES

Oddly enough, theorems seem to have a less formal structure than definitions – or, at least, there is more variety in these structures. Perhaps this is because theorems could, in principle, be any (non-obvious) mathematical statement which can be demonstrated to be *true*. Nevertheless, to understand at least the statement, for most theorems, the following are good things to keep in mind.

Hypotheses/Context: Most theorems apply only to objects or situations only with certain characteristics – these are the *hypotheses* and/or *context*. It is vital to know them, and to know

- (1) how to check their validity in particular situations;
- (2) an example (or several; the simpler the better) which satisfies all the hypotheses; and
- (3) counterexamples to the conclusions of theorem in case any one of the hypotheses is left off.

Conclusion[s]: What is the result, *precisely*: in particular, all technical terms and notation used must be familiar, and all statements completely clear and precise.

Variables: **All** variables must be, as always, defined and quantified.

“The Big Idea”: Beyond the precise statement, it is nice to have a less formal way of describing or thinking about what the theorem is saying.

The following is an example of this kind of analysis:

Theorem. *A matrix is invertible if and only if 0 is not an eigenvalue.*

- (1) *[Context:]* Fix an $n \in \mathbb{N}$, and an $n \times n$ matrix A .
- (2) *[Conclusion:]* Define statements $\mathcal{A} = “A \text{ is invertible}”$ and $\mathcal{B} = “0 \text{ is not an eigenvalue of } A”$. Then this theorem is saying that $\mathcal{A} \Leftrightarrow \mathcal{B}$.
- (3) *[Examples:]* The identity matrix is certainly invertible, and has only the eigenvalue 1.
- (4) *[Checking hypotheses:]* We can check if A is invertible by computing its determinant. We can compute the eigenvalues by factoring $\text{char}(A)$. Both of these are quite easy if A is 2×2 , for example.
- (5) *[Examples with different hypotheses:]* The zero matrix is certainly singular, and has only the eigenvalue 0.
- (6) *[Big Idea:]* This theorem seems to be saying that the presence of a zero eigenvalue is something like having a zero determinant – it tells us if the matrix is invertible. Also, multiplication by a matrix acts like multiplication by the eigenvalue λ , at least on the corresponding eigenvector, so it stands to reason that this is invertible only if there is no zero eigenvalue – multiplication by zero being the only scalar multiplication which is not invertible.

A future handout will have a form or outline for analyzing theorems (like the one for definitions).

Here are theorems/results for which this kind of analysis is appropriate:

- **Gaussian (or Gauss-Jordan) Elimination**
- **The Dimension Theorem**
- **The Rank-Nullity Theorem**
- **Theorem relating matrices and linear transformations**
- **The determinant is well-defined [or Laplace’s Expansion Theorem]**
- **Theorems on different ways to compute the rank of a matrix**
- **Theorem relating eigenvalues to roots of the characteristic polynomial**
- **The Fundamental Theorem of Invertible Matrices**
- **Theorem[s] on diagonalizability and eigenvalues**

<i>[Context]</i>		<i>[Type]</i>
<i>[Def]</i>		
<i>[Idea]</i>		
<i>[e.g.]</i>		
<i>[c.f.]</i>		

<i>[Context]</i>		<i>[Type]</i>
<i>[Def]</i>		
<i>[Idea]</i>		
<i>[e.g.]</i>		
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