

CSU-PUEBLO MATH 425
SPRING TERM 2010
A PROOF OF A SPECIAL CASE OF
THE CAUCHY-GOURSAT THEOREM

The Cauchy-Goursat Theorem: If $f(z)$ is analytic on and inside a simple closed contour C then it follows:

$$\int_C f(z) dz = 0 \quad .$$

Proof (part): We make two simplifying assumptions:

- C is the rectangle in the complex plane whose shadow on the x -axis is the interval $[a, b]$ and whose shadow on the y -axis is the interval $[c, d]$ — i.e., the corners of this rectangle are $a + ic$, $b + ic$, $b + id$ and $a + id$;
- the real and imaginary parts of $f(z)$ have integrable first order partial derivatives.

We parametrize the four sides of this rectangle as:

$$\begin{aligned}\gamma_1 : [a, b] &\rightarrow \mathbb{C} : t \mapsto t + ic \\ \gamma_2 : [c, d] &\rightarrow \mathbb{C} : t \mapsto b + it \\ \gamma_3 : [a, b] &\rightarrow \mathbb{C} : t \mapsto (a + b - t) + id \\ \gamma_4 : [c, d] &\rightarrow \mathbb{C} : t \mapsto a + i(c + d - t)\end{aligned}$$

Note the following expressions for the derivatives:

$$\begin{aligned}\gamma_1'(t) &= 1 \\ \gamma_2'(t) &= i \\ \gamma_3'(t) &= -1 \\ \gamma_4'(t) &= -i\end{aligned}$$

Then we write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and we have

$$\begin{aligned}\int_C f(z) dz &= \int_a^b (u(t, c) + iv(t, c)) dt + \int_c^d (u(b, t) + iv(b, t)) i dt \\ &\quad + \int_a^b (u((a + b - t), d) + iv((a + b - t), d)) (-1) dt \\ &\quad + \int_c^d (u(a, (c + d - t)) + iv(a, (c + d - t))) (-i) dt\end{aligned}$$

Now look at the piece $\int_a^b (u((a + b - t), d) + iv((a + b - t), d)) (-1) dt$. If we change variables to $s = (a + b - t)$ it becomes $\int_b^a (u(s, d) + iv(s, d)) ds$. Then switching the endpoints of the integration and calling the integration variable t again (names of variables don't matter!), we see that this piece actually equals $-\int_a^b (u(t, d) + iv(t, d)) dt$. The same analysis makes the last term above become $-i \int_c^d (u(a, t) + iv(a, t)) dt$. Therefore, putting integrals together, we get

$$\begin{aligned}\int_C f(z) dz &= \int_a^b ((u(t, c) - u(t, d)) + i(v(t, c) - v(t, d))) dt \\ &\quad + i \int_c^d ((u(b, t) - u(a, t)) + i(v(b, t) - v(a, t))) dt \quad .\end{aligned}$$

But now we use the Fundamental Theorem of Calculus for each of the pieces like

$$u(t, c) - u(t, d) = - \int_c^d u_y(t, s) \, ds$$

in order to write

$$\int_C f(z) \, dz = \int_a^b \int_c^d -(u_y(t, s) + iv_y(t, s)) \, ds dt + i \int_c^d \int_a^b (u_x(s, t) + iv_x(s, t)) \, ds dt \quad .$$

We switch the order of integrals and the names of the variables s and t in the second term above, and combine terms to get

$$\int_C f(z) \, dz = \int_a^b \int_c^d (u_y(t, s) + v_x(t, s)) + i(u_x(t, s) - v_y(t, s)) \, ds dt \quad .$$

But the Cauchy-Riemann equations say exactly that the integrand on the right side here has vanishing real and imaginary parts, and since integrating the constant function 0 over the rectangle $[a, b] \times [c, d]$ yields 0, we have proven this special case of the Cauchy-Goursat Theorem. $\square$