# WHAT MAKES A GOOD DEFINITION AND AN UNDERSTANDING THEREOF IN MATHEMATICS

JONATHAN A. PORITZ

We shall consider definitions of two types of mathematical terms, those for

**mathematical objects:** these are the nouns of the mathematical language, such as *integer, vector, rank, etc.*; or for

**properties of mathematical objects:** which are the adjectives in our language, such as *even, odd, prime, consistent, homogeneous, etc.*

## 1. Mathematical Objects

A good definition has a number of features or components. As you get more experienced, you may leave some of these out when they should be clear to your intended audience from the context... but to start with you should **always include all of them**. One way to be sure of including all required features is to use the forms as below and on the following pages.

Beyond the logically required parts of a definition, there are a few things which you must know in order to understand fully the defined term. There are fields for these in the forms below, as well.

In the forms, the meaning of the boxes is as follows, where an underlined term is logically required and one without underlining is [merely] necessary for complete understanding of the term.

*Context*: The environment in which this definition makes sense, such as the universe of possible values for an unbound variable that exists in the definition. The *Context* field can be empty for the simplest definitions.

*Term*: The term being defined.

*Type*: What kind of object it is – a real number, a set, a matrix, *etc.* The *Type* field will be empty for definitions of **mathematical properites**, the adjectives of the language of mathematics, but is required for **mathematical objects**, the nouns of this language.

*Def*: A complete, formal definition of the object – make sure all variables in this definition are either the thing being defined, part of the *Context*, or have associated **quantifiers**.

*Idea:* A description in your own words of what the *essential idea* of this term is.

*E.g.:* A complete, careful, explicit example (or more than one) of this term.

*C.f.:* Names or descriptions of one or more theorems, constructions, algorithms, or other definitions in which this term is used.

*Well-def'd:* What would need to be true for this object to be well-defined (if anything); are there special cases which require special treatment?

The following is an example:

| | | |
|---|---|---|
| *[Context]* Given $k, n \in \mathbb{N}$ and $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ | *[Term]* we define $\underline{\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_k)}$ to be | *[Type]* the subset of $\mathbb{R}^n$ |
| *[Def]* given by $\{c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$ . | | |
| *[Idea]* The Span is the set of all linear combinations of the given vectors. | | |
| *[e.g.]* $\mathrm{Span}(\vec{v})$ is the line through the origin and $\vec{v}$; $\mathrm{Span}(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix})$ is the $x$-$y$ plane in $\mathbb{R}^3$ ... | | |
| *[c.f.]* **Prop:** $\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_k)$ always contains the vectors $\vec{0}, \vec{v}_1, \ldots,$ and $\vec{v}_k$ <br> **Thrm:** A Span is always a vector subspace of $\mathbb{R}^n$ | | *[Well-def'd]* nothing to check |

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## 2. Properties of Mathematical Objects

The definition of a property of mathematical objects is just like the definition of a mathematical object where the *Type* is {TRUE or FALSE} – such a type is called a *Boolean* (named after the English mathematician George Boole who studied mathematical logic).

Thus an example of a definition of a property might be

| | | |
|---|---|---|
| *[Context]* Given $p \in \mathbb{N}$ s.t. $p > 1$, | *[Term]* we say $p$ is <u>prime</u> | *[Type]* BOOLEAN |
| *[Def]* if $\forall n \in \mathbb{N}$ if $n$ is a factor of $p$, then $n = 1$ or $n = p$. | | |
| *[Idea]* A number is prime if it has no "non-trivial" factors. | | |
| *[e.g.]* 2 is the only even prime; 17 is prime; $2^{43,112,609} - 1$ is the largest known prime *[in January 2012]*. | | |
| *[c.f.]* **Thrm [Euclid]:** There are an infinite number of prime numbers. **Thm:** Every natural number can be written as a product of primes. | | *[Well-def'd]* notice 1 is not prime! |

## 3. Practice

Now make some definitions of your own:

| | | |
|---|---|---|
| *[Context]* | *[Term]* | *[Type]* |
| *[Def]* | | |
| *[Idea]* | | |
| *[e.g.]* | | |
| *[c.f.]* | | *[Well-def'd]* |

| | | |
|---|---|---|
| *[Context]* | *[Term]* | *[Type]* |
| *[Def]* | | |
| *[Idea]* | | |
| *[e.g.]* | | |
| *[c.f.]* | | *[Well-def'd]* |

| | | |
|---|---|---|
| *[Context]* | *[Term]* | *[Type]* |
| *[Def]* | | |
| *[Idea]* | | |
| *[e.g.]* | | |
| *[c.f.]* | | *[Well-def'd]* |

| | | |
|---|---|---|
| *[Context]* | *[Term]* | *[Type]* |
| *[Def]* | | |
| *[Idea]* | | |
| *[e.g.]* | | |
| *[c.f.]* | | *[Well-def'd]* |

| | | |
|---|---|---|
| *[Context]* | *[Term]* | *[Type]* |
| *[Def]* | | |
| *[Idea]* | | |
| *[e.g.]* | | |
| *[c.f.]* | | *[Well-def'd]* |