

Perelman Proves Poincaré

Jonathan A. Poritz

`jonathan.poritz@gmail.com`
`poritz.net/jonathan`

Department of Mathematics & Physics
Colorado State University, Pueblo
2200 Bonforte Blvd.
Pueblo, CO 81001-4901

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The Poincaré Conjecture



Henri Poincaré, 1904:

Every **compact**,
connected,
simply-connected,
3-manifold,
is the **3-sphere**.

Manifolds [Roughly]

$\mathbb{R}^n$ is the set $\{(x_1, \dots, x_n)\}$ of n -tuples of real numbers. *E.g.:*

- $\mathbb{R}^1$ is the usual “number line”
- $\mathbb{R}^2$ is Descartes’ version of the Euclidean plane
- $\mathbb{R}^3$ is “three-dimensional space”

An n -**manifold** M [*without boundary*] is a set such around every point of M there can be found a sufficiently small neighborhood which looks like part of $\mathbb{R}^n$. *E.g., when $n = 2$:*

- all of $\mathbb{R}^2$
- an open disk in $\mathbb{R}^2$ (but **not** a closed disk)
- *punctured* $\mathbb{R}^2$ (or a punctured disk)
- several of the above examples (perhaps repeated copies of one) next to each other

Topology

A **metric** d on a space [set] X is a function $d : X \times X \rightarrow \mathbb{R}$ satisfying:

- $\forall x, y \in X, d(x, y) = d(y, x)$ [symmetry]
- $\forall x, y \in X, d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
- $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$ [the triangle inequality]

A **topology** on a space [set] X is a collection $\mathcal{O}$ of subsets of X (called the **open sets** of X) satisfying:

- $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$
- $\forall A, B \in \mathcal{O}, A \cap B \in \mathcal{O}$ [closed under finite intersections]
- $\forall \mathcal{C} \subseteq \mathcal{O}$ of open sets, $\bigcup_{C \in \mathcal{C}} C \in \mathcal{O}$ [closed under arbitrary unions]

A **topological space** is a space X together with a topology on X .

Metrics induce topologies, but not every topology is “metrizable.”

A **continuous function** $f : X \rightarrow Y$ between topological spaces is one for which f^{-1} of every open set in Y is open in X .

Connected & Compact

A topological space X is **connected** if you can get from any point of X to any other point by an continuous path. *E.g.:*

- Any of the above *single* examples.
- Any manifold which doesn't fall into pieces when you pick it up.

A topological space X is **compact** if any infinite sequence of points from X “piles up” at *[has a limit point which is]* some point of X . *E.g., when $n = 2$:*

- the 2-sphere and its evil twin $\mathbb{RP}^2$
- the torus T^2 and its evil twin the Klein bottle $\tilde{T}^2$
- the higher genus Riemann surfaces Σ_g , for $g > 1$, and their evil twins $\tilde{\Sigma}_g$

But **not** all of $\mathbb{R}^2$ or a Riemann surface of infinite genus.

Homotopy

Two maps $f_0, f_1 : X \rightarrow Y$ between topological spaces are **homotopic** if there exists a continuous map $F : X \times [0, 1] \rightarrow Y$ such that $\forall x \in X$, $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We write $f_0 \simeq f_1$.

Topological spaces X and Y are **homotopic** if there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$.

Homotopy of topological spaces is a very weak notion: it merely means that one can be pulled and stretched and twisted without tearing [*continuously deformed*] to become the other. *E.g.*, Topologists famously confuse their coffee cups with their doughnuts because they are homotopic.

[A stronger notion is **homeomorphism**: a continuous map with a continuous inverse.]

The Fundamental Group

Start with a connected topological space X and one particular point p of X . Look at the set of all continuous loops on X which start and end at p . Consider two such loops to be “the same” if they are homotopic. The set of distinct loops that remains is called the **fundamental group of M** , written $\pi_1(X)$.

If the only loop (up to homotopy, remember) in $\pi_1(X)$ is the trivial loop (which starts, ends, and in fact spends its whole time at p), then we say X is **simply-connected**. *Examples:*

- $\mathbb{R}^2$ and S^2 are simply-connected
- $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$
- $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$
- $\pi_1(\Sigma_g)$ is infinite but has a simple description

Equivalently, X is simply-connected if any closed loop based at p can be continuously deformed to the trivial loop at p .

Smooth Manifolds and other additional structures

To complete the definition of a **smooth** manifold M , we require that the maps from neighborhoods of M to $\mathbb{R}^n$ satisfy the compatibility condition that on overlaps the compose to make an infinitely differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. This enables us to take derivatives of functions on manifolds, to solve differential equations, and, with some additional structure (a notion of the length of a tangent vector to a smooth curve on a manifold (called a **Riemann metric**)) to compute the lengths of curves.

On a manifold endowed with a Riemannian metric (called a **Riemann manifold**), we say a curve is a **geodesic** if it is the shortest curve connecting every pair of its points.

Another kind of additional structure was emphasized by the German mathematician Felix Klein in his *Erlangenprogramm*, to wit:

Transformation Groups

A collection G of transformations of a manifold M is called a **transformation group acting on M** if:

- G contains the trivial (= "identity") transformation;
- whenever f and g are in G , so is their composition $f \circ g$;
- whenever f is in G , so is its inverse f^{-1} .

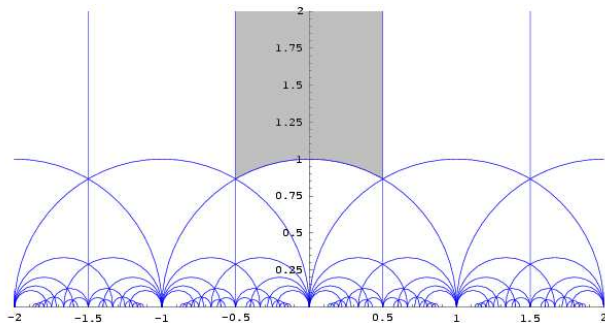
In the above situation, we can define the **quotient** M/G of M by the action of G as the space M again, but now with a new notion that two points p and q will be considered "the same" if $q = f(p)$ for some f in G .

E.g., $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

$$\Sigma_g = \mathcal{H}^2 / \Gamma_g \quad [What's that?]$$

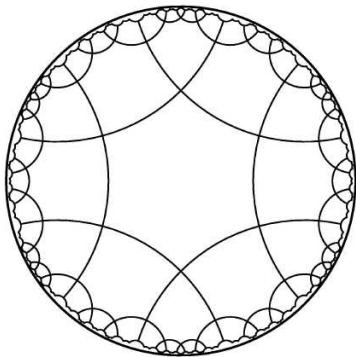
The Hyperbolic Plane

As a set, the **hyperbolic plane** $\mathcal{H}^2$ is the upper half of $\mathbb{R}^2$. It is given a metric so that the following are all straight lines



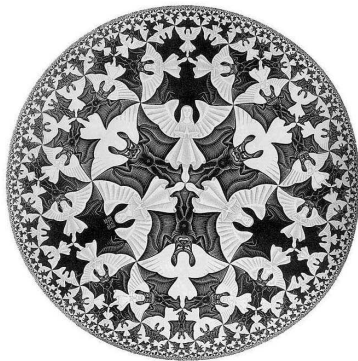
The Hyperbolic Disk

The hyperbolic plane is isometric to the **hyperbolic disk** which consists of the interior of the unit disk in $\mathbb{R}^2$, endowed with a metric so that the straight lines are now



Angels and Demons

[Was that last picture reminiscent of



?]

Curvature

S^2 is said to have **positive curvature**, because when it sits in $\mathbb{R}^3$, at each point there are two perpendicular directions which both curve to the same side of the tangent plane. Triangles on S^2 have interior angle sums which always **exceed** 180° .

T^2 is said to be **flat**, because its intrinsic geometry is classically Euclidean (although it cannot be embedded in $\mathbb{R}^3$ with this metric). Triangles on T^2 have interior angle sums which always **equal** 180° .

Σ_g , for any $g > 1$, is said to have **negative curvature** when it is built by identifications of sides of polygons in $\mathcal{H}^2$; it cannot be embedded in $\mathbb{R}^3$ with this metric. Triangles on Σ_g have interior angle sums which are always **less than** 180° .

Actually, all of these are **constant curvature** manifolds.

Classification in Two Dimensions

All compact, connected 2-dimensions are homotopic (so, “the same as”, to a topologist; actually, they are also *homeomorphic* and even *diffeomorphic*) to one of the manifolds

$$S^2, \mathbb{RP}^2, T^2, \tilde{T}^2, \Sigma_g, \tilde{\Sigma}_g$$

where g is any positive integer.

This is essentially something known in complex analysis as **The Uniformization Theorem**. It can be proven in several ways, for example by putting some metric on the 2-manifold and then solving a differential equation (similar to the heat equation) to reform the metric until it has constant curvature, then using the very simple geometry of constant curvature spaces.

Wait, what? – Proof strategies

A topology is very little structure on a space. There just isn't enough to get your fingernails into in order to prove theorems. [This is also perhaps why it is hard to prove theorems about prime numbers.]

One strategy to prove theorems on a topological space (or smooth manifold) then is to endow it with additional structure (like a Riemann metric), then to reform that structure (maybe by solving a differential equation), so that in the end it is such a nice and rich structure many things can be calculated about it.

An equation used in this way is one similar to the *heat equation*, giving

- for $n = 1$, the “curve shortening flow” ... curves get round (and short)
- for $n = 2$, this gives in the limit a constant curvature metric and therefore geometric structure on Σ_g
- for $n = 3$, Hamilton and then Perelman studied the **Ricci flow**.

Three dimensions

There is a 3-dimensional analogue of hyperbolic space defined as follows: $\mathcal{H}^3$ is the set of points (x, y, z) in $\mathbb{R}^3$ satisfying $z > 0$ (the “upper half-space”). To find the distance between two points in $\mathcal{H}^3$, or to connect them with a straight line, consider the vertical half-plane in $\mathbb{R}^3$ containing the points. The distance between the points, and the straight line connecting them, will be the distance and line in that half-plane thought of as $\mathcal{H}^2$.

There is also a spherical geometry S^3 (the usual round sphere in $\mathbb{R}^4$) and a flat geometry $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ (or a cube with opposite faces identified) in three dimensions.

In fact, there are **five more exotic** geometries in 3-dimensions. All together these geometries can be called “Thurston’s 8-fold Path”.

Thurston Geometrization



Wild Bill Thurston, late 1970's:

Every 3-manifold can be canonically decomposed into pieces, each of which admits a geometric structure from the 8-fold path.

Perelman's Proof



Grigori (Grisha) Perelman, late 2003/4:

Thurston's Geometrization Conjecture
is true

so also the Poincaré Conjecture
is true

Higher Dimensions

FYI: the situation is quite different in higher dimensions than three.

First, in four dimensions (think: spacetimes?) the situation is very complex and while great work was done in the 1980s, yielding very surprising results, this area has somewhat stalled.

Oddly enough, dimensions even higher, so 5 and up, the situation is much, much simpler. There is so much room in higher-dimensional spaces that the analogue of the Poincaré conjecture was proven around 40 years ago.

Morals

- ❶ **Mathematical Strategy:** Often it is hard to prove theorems about mathematical objects with very simple definitions. If so, it can help to add extra structure and then to solve an equation for a particularly nice case of that structure.
- ❷ Physics can loan us nice differential equations – e.g., the heat equation and the Ricci flow (which is related to General Relativity).
- ❸ **Thurston's Geometrization Conjecture is true!** It should be the start of a very exciting period in three-dimensional geometry....
- ❹ Grish Perelman is a very unusual and admirable character (e.g., he turned down the Fields Medal, a \$1million prize from the Clay Institute, and jobs at famous universities because “the work is what is important; I did it and now it should just stand on its own... and I don't need the money or the job”). See various recent books about him if you are interested, such as the one by Masha Gessen.