

Math 495

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The Riemann Hypothesis: The #1 Problem in Mathematics



Georg Friedrich Bernhard Riemann 1826-1866

One of 6 Million Dollar prize problems (Clay Mathematics Institute)

To literally win a million dollars, you first need to be able to answer “What is the Riemann zeta function?” Here it is (version 1):

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

It's given as a series, but not a power series. (In calculus II, you might have called it a p -series, $\sum \frac{1}{n^p}$, but we use s rather than p – do you remember when it converges? diverges?)

So

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \textit{Euler's?}$$

[zeta(2) is the sum of the reciprocals of the perfect squares...from calc II, you might say it's a p-series with p=2].

Why is it – the Riemann zeta function – important?

The zeta function encodes everything about prime numbers.

If you haven't had number theory, let's take a step back to remember the basics about primes, and why they're important. We'll also need to review some basics about complex numbers, and also some ideas from calculus about series. Then we can get back to the Riemann zeta function.

1-Quick prime number review

Is 17 a prime? Is 23 a prime? Is 49 a prime? Is 91 a prime?

$$\begin{array}{r} 13 \\ \times 7 \\ \hline ?? \end{array}$$

Is 221 a prime?

$$\begin{array}{r}
 13 \\
 \times 17 \\
 \hline
 91 \\
 13 \\
 \hline
 221
 \end{array}$$

Every whole number can be expressed as a unique product of primes – even if you have not had number theory, you learned about this maybe in 3rd or 4th grade:

$$\begin{aligned}
 36 &= 2 \times 2 \times 3 \times 3 \\
 91 &= 13 \times 7 \\
 100 &= 2 \times 2 \times 5 \times 5 \\
 101 &= 101 \text{ (101 is a prime)}
 \end{aligned}$$

Short detour: a few beautiful formulae – Leibniz's and Viete's for π

Viete's late 16th century formula for pi (and infinite product)

$$\frac{\pi}{2} = \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2 + \sqrt{2}}} \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \dots$$

Leibniz's late 17th century formula for pi (an infinite series)

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right)$$

The Basel problem solved by Euler (early 18th century) (an infinite series)

$$\pi^2 = 6 \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots \right)$$

or

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$$

Do you remember this function?

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

So

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \text{Euler's?}$$

i.e. $\zeta(2) = \pi^2/6$; in the notation we've used,

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{is one value of Riemann's zeta function.}$$

So

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \text{Euler's?} = \frac{\pi^2}{6}$$

Zeta(2) involves pi in a nice strange way...

Can we go back to Riemann and the #1 problem in mathematics? Not yet...

2-Complex numbers (first, practice with some algebra)

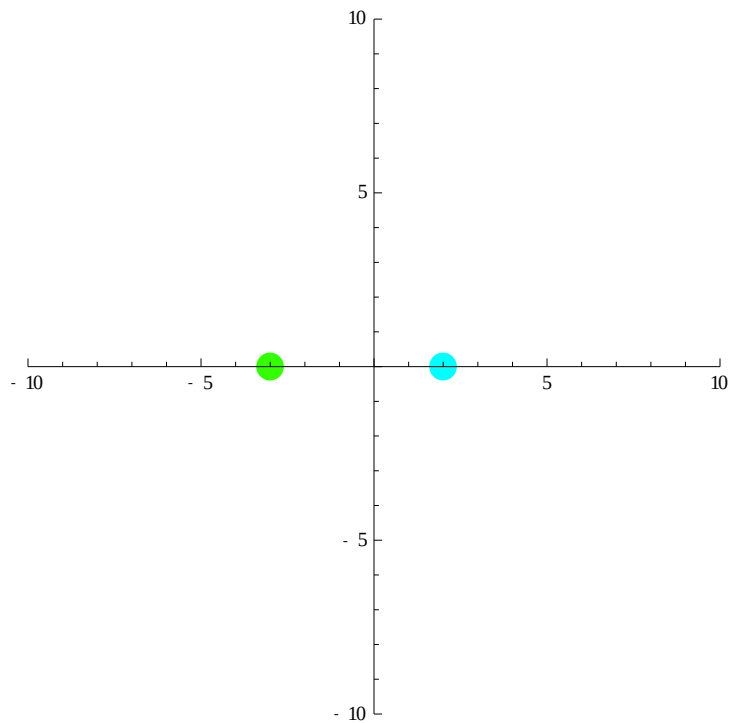
Solve $x^2+x-6=0$ for x

$$(x+3)(x-2)=0$$

So

$$x=-3 \text{ or } x=2$$

Let's use the complex plane to plot the two zeroes:



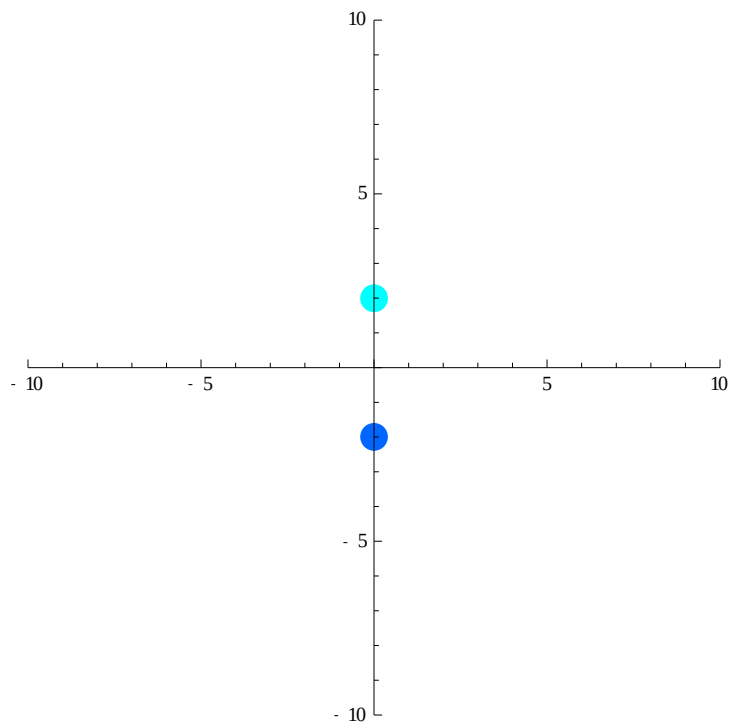
Solve $x^2 + 4 = 0$ for x ?

We usually use 'z' for a complex variable (instead of x).

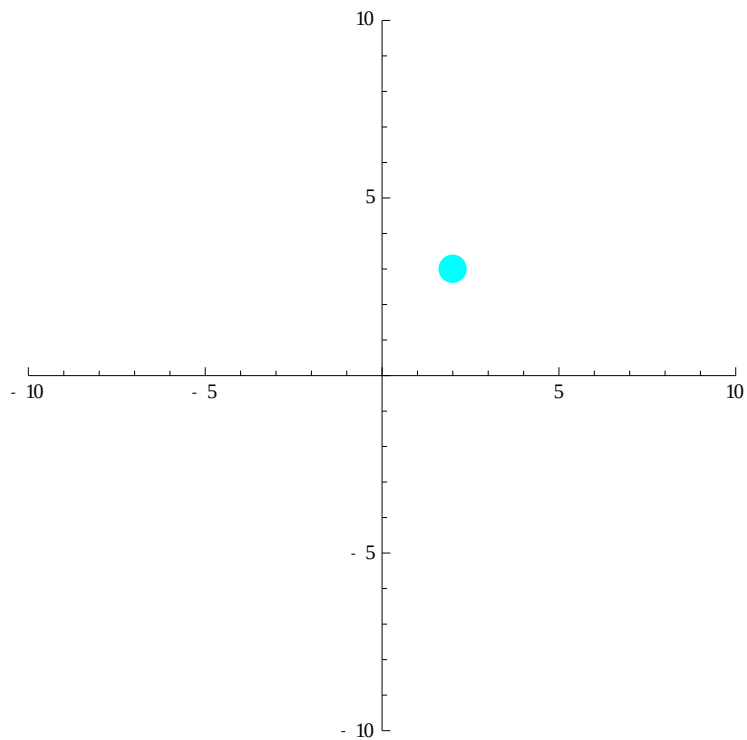
So solve $z^2 + 4 = 0$ for z ?

$z = 2i$ or $z = -2i$?

How do we plot $2i$ and $-2i$? Like so:



What number is depicted below?



3 - Calculus II and series: “analytic continuation”

Consider the function

$$f(s) = 1 - s + s^2 - s^3 + \dots$$

Somewhat familiar? If you forgot what it is, notice that it contains itself:

$$f(s) = 1 - s (1 - s + s^2 - s^3 + \dots),$$

i.e.

$$f = 1 - s (f)$$

So

$$f + s f = 1,$$

i.e.

$$f (1+s) = 1,$$

so

$$f = 1/(1+s)$$

and thus

$$f(s) = \frac{1}{1+s}$$

In calc II, you know that actually the original f , the original series, is only convergent for $|s| < 1$ (e.g. by the ratio test).

But the second version, $f(s) = 1/(1+s)$, is actually well-defined for all s except $s=-1$. In fact, if we say s could be a complex #, then it's convergent in the entire complex plane except for $s = -1$.

We say the complex function $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic everywhere except at $s=-1$, and it has a simple pole at $s = -1$.

Bottom line:

So the original function, given by the series, and convergent only for $|s| < 1$, has an analytic continuation to a function defined in the entire complex plane except at $s = -1$.

So the original series makes no sense if, say, $s=2$; but the analytically continued version makes sense

Original series at $s=2$: $f(s) = 1 - s + s^2 - s^3 + \dots$, so

$$f(2) = 1 - 2 + 4 - 8 + 16 - \dots = ?!?!?$$

Value of analytic continuation at $s=2$:

$$f(s) = 1/(1+s) = 1/3$$

So while it's not correct to say it, in some sense

$$“1 - 2 + 4 - 8 + \dots = 1/3” (!)$$

Aside #2: We had $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$. What is 2^s ?!?

One way to compute 2^s : it's $(e^{\ln 2})^s$, i.e. $e^{.693147\dots s}$

And you know how to compute e^z ; it's $1 + z + z^2/2! + z^3/3! + z^4/4! + \dots$

So let's compute the cube root of 2 this way:

We need $e^{.231049\dots}$, i.e. $1 + .231049 + .231049^2/2 + .231049^3/6 + .231049^4/24 + \dots = 1.259915\dots$ (and if we carry more terms, we get that the cube root is really 1.25992...)

Note that $e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \dots$ is defined for all z , real or complex, i.e. the exponential function $f(z)=e^z$ is 'entire' – it's analytic in the entire complex plane.

So we could compute 2^{3+4i} the same way: just plug in $z=.693147(3+4i)$ into the series for e^z

So the function $f(s)=2^s$ is also entire; it's defined in the entire complex plane. So is the function $f(s)=1/2^s$.

Back to the Riemann zeta function

For $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$

Clearly each term can be defined for any complex number s .

But from calc II, we know we have a problem if $s=1$ (think of p-series; or directly $\zeta(1) = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$). So we know that ζ has at least one problem location, at $s=1$.

But the zeta function can be analytically continued.

Turns out: it can be analytically continued everywhere in the complex plane, even to $s=0$!?, except it has a 'simple pole' at $s=1$.

So $\zeta(0) = 1+1/2^0 + 1/3^0 + 1/4^0 + \dots$ should be $1+1+1+1+\dots$; but the analytic continuation satisfies $\zeta(0) = -1/2$ (!)

Initial connection of Riemann's zeta function and the primes

Consider the geometric series

$$(1+1/2^2+1/2^4+1/2^6+1/2^8+\dots) \text{ or } (1+1/4+1/16+1/64+\dots);$$

we know its value is $1/(1-1/2^2)$, or $4/3$ (...you could use our 'f' function from a few slides ago).

Similarly consider

$$(1+1/3^2+1/3^4+1/3^6+\dots) \text{ or } (1+1/9+1/81+1/729+\dots)$$

We know its value is $1/(1-1/3^2)$, or $9/8$.

Similarly $(1+1/5^2+1/5^4+1/5^6+\dots)$ is $1/(1-1/5^2)$ or $25/24$.

Do this for each prime 2, 3, 5, 7, 11, ...

Multiply these together:

$$\begin{aligned} & (1+1/2^2+1/2^4+1/2^6+\dots)(1+1/3^2+1/3^4+1/3^6+\dots)(1+1/5^2+1/5^4+1/5^6+\dots)(1+1/7^2+1/7^4+\dots) \\ &= 1+1/2^2+1/3^2+1/4^2+1/5^2+1/6^2+1/7^2+1/8^2+\dots \end{aligned}$$

i.e. we get

$$(1/(1-1/2^2)) (1/(1-1/3^2)) (1/(1-1/5^2)) (1/(1-1/7^2)) = \zeta(2)$$

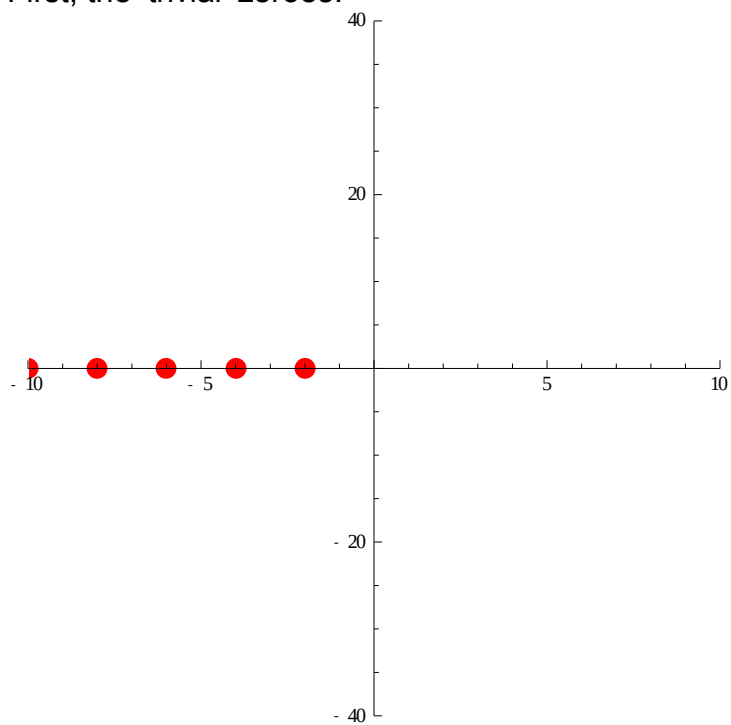
In general

$$\prod_{\text{primes } p} \frac{1}{(1-1/p^s)} = 1+1/2^s + 1/3^s + 1/4^s + 1/5^s + 1/6^s + \dots = \zeta(s)$$

at least for real part of $s>1$ (where the greek upper case Pi means product). This is an example of an Euler product. Trading infinite products with infinite series is commonplace in analytic number theory (L-functions, etc).

The zeroes of the Riemann zeta function

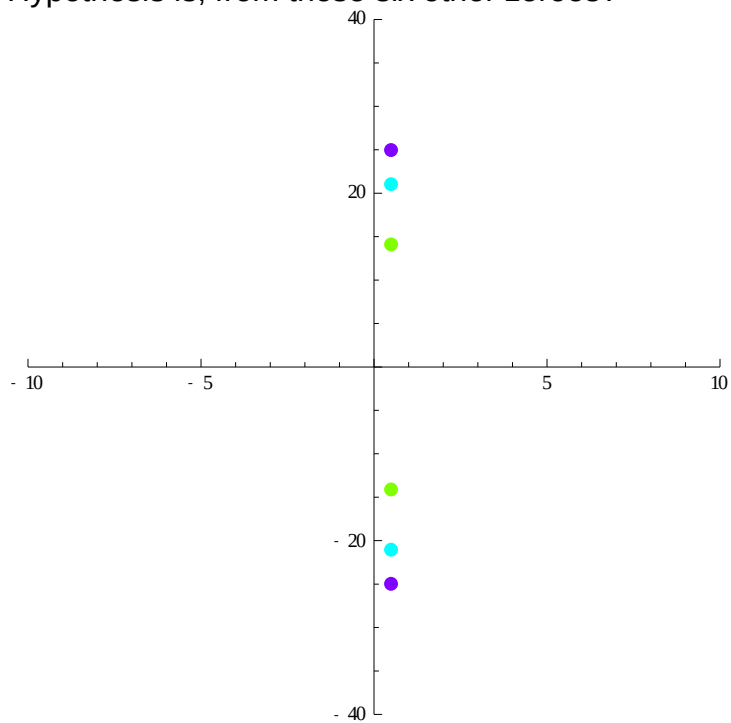
First, the 'trivial' zeroes:



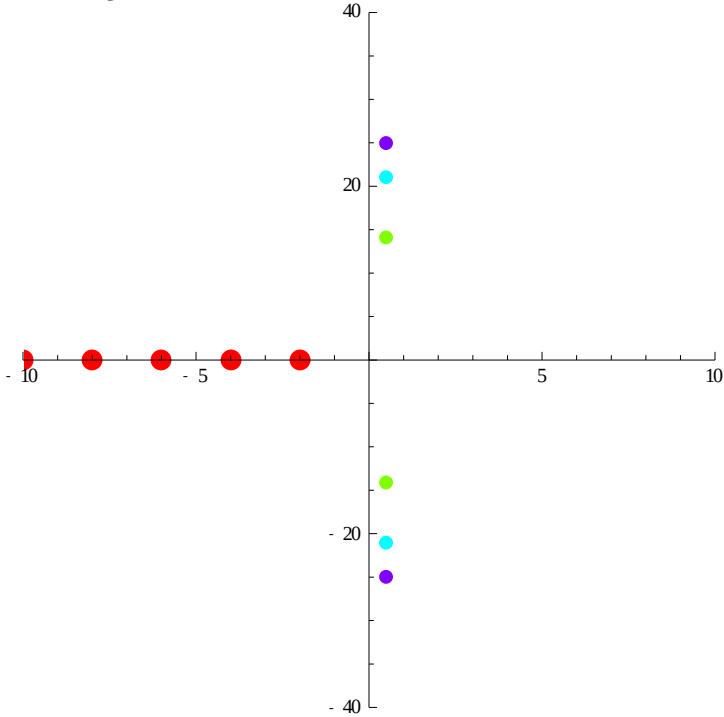
Where are these?

So these zeroes are at $z = -2, -4, -6, -8, -10, \dots$

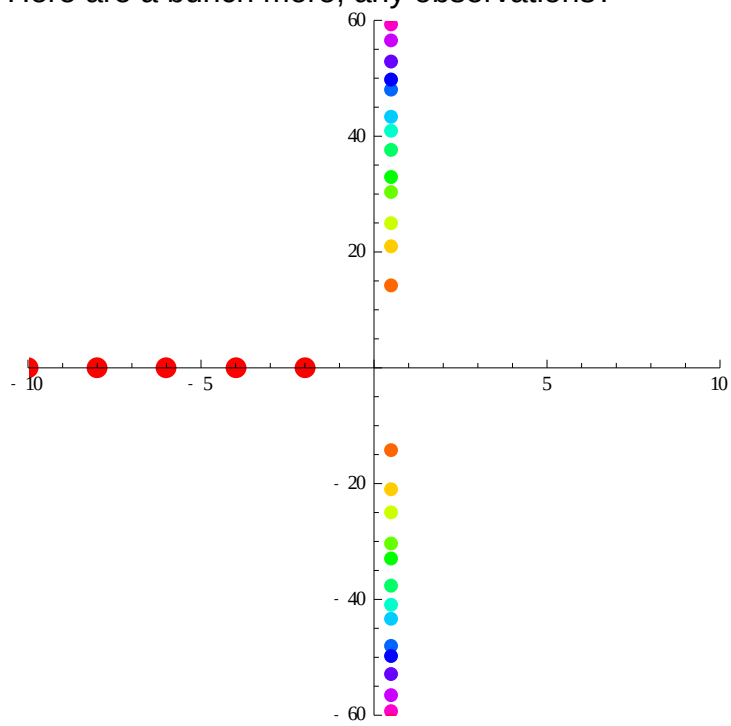
Next, Riemann found the first other six zeroes; can you guess what the Riemann Hypothesis is, from these six other zeroes?



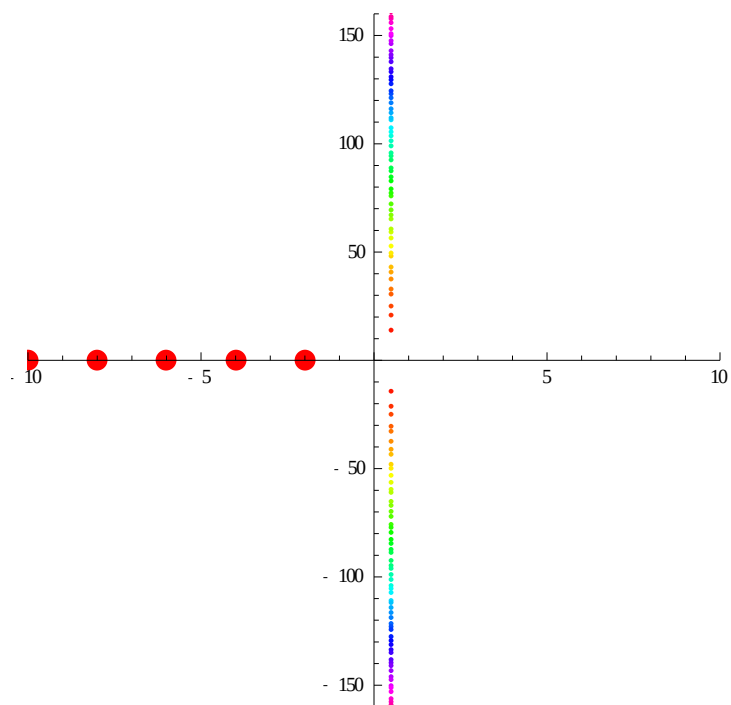
So altogether here are some of the zeroes:



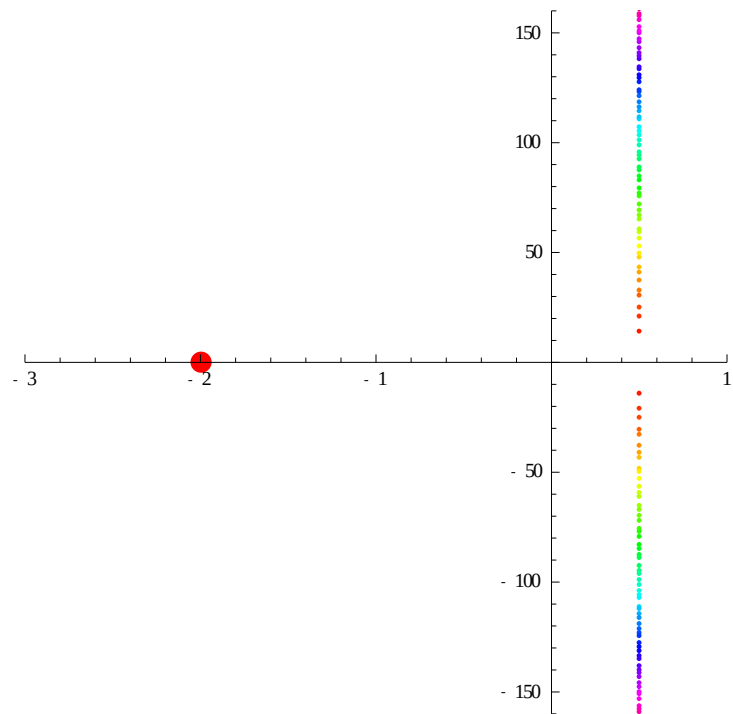
Here are a bunch more; any observations?

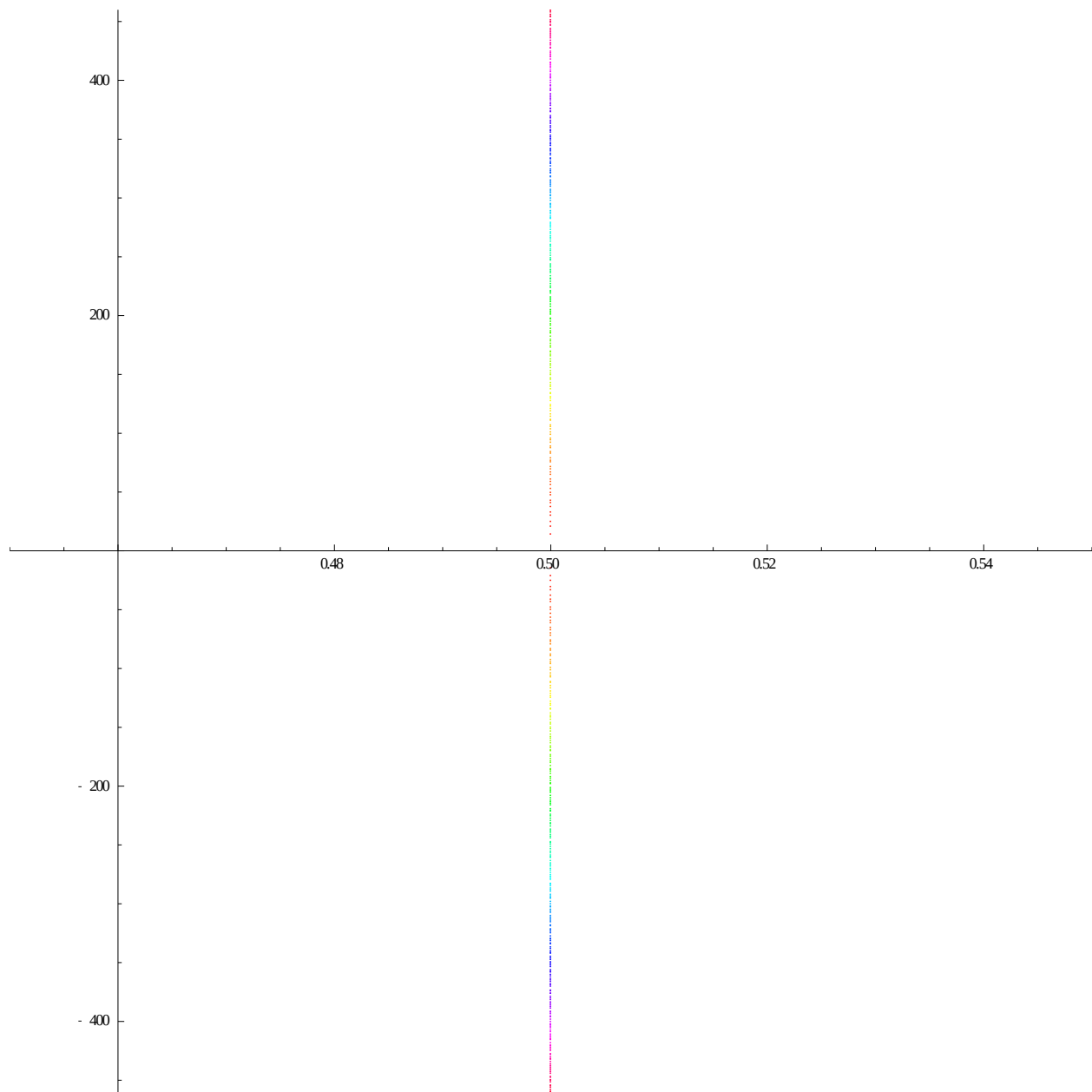


And even more:



Let's blow it up:





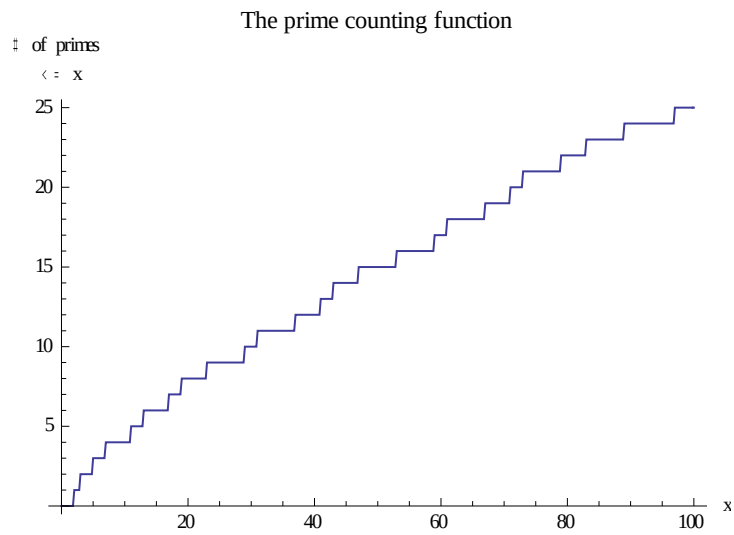
By now, we have guessed what Riemann hypothesized: that for all the nontrivial zeroes, they must satisfy that their real part is exactly $\frac{1}{2}$. (Remember Riemann only computed 3 pairs of zeroes explicitly when he made his conjecture.)

What does this have to do with prime #s?

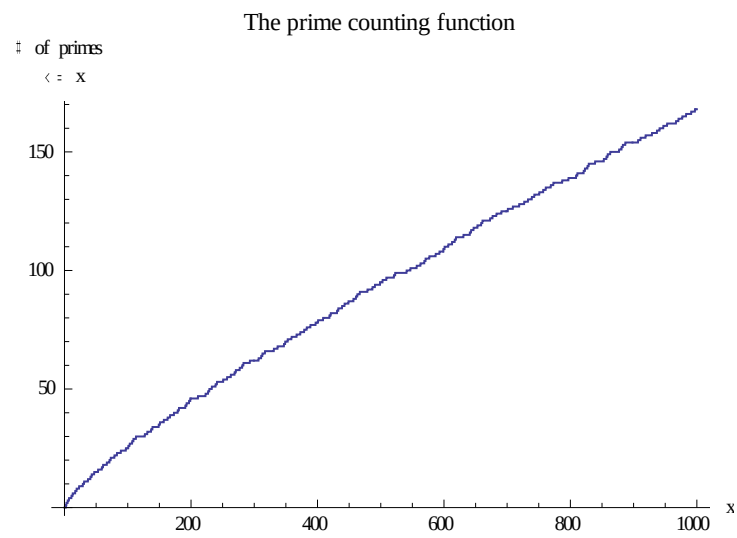
The prime counting function

How many primes are less than or equal to 5?

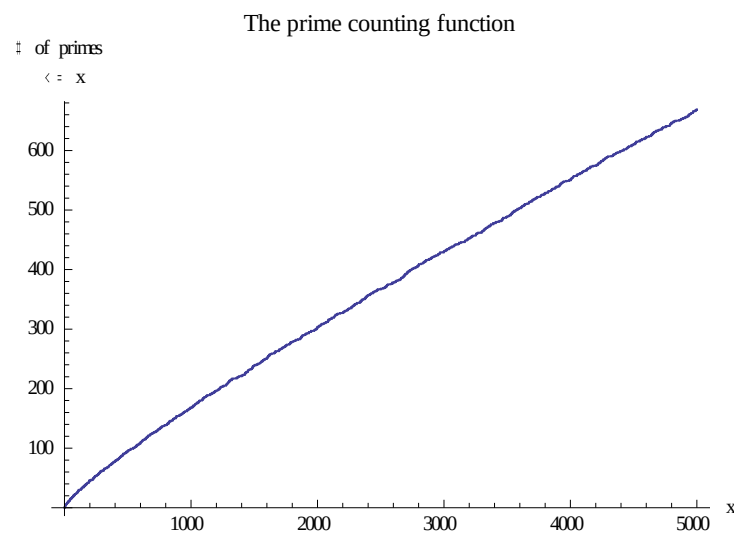
How many primes are less than or equal to 10?



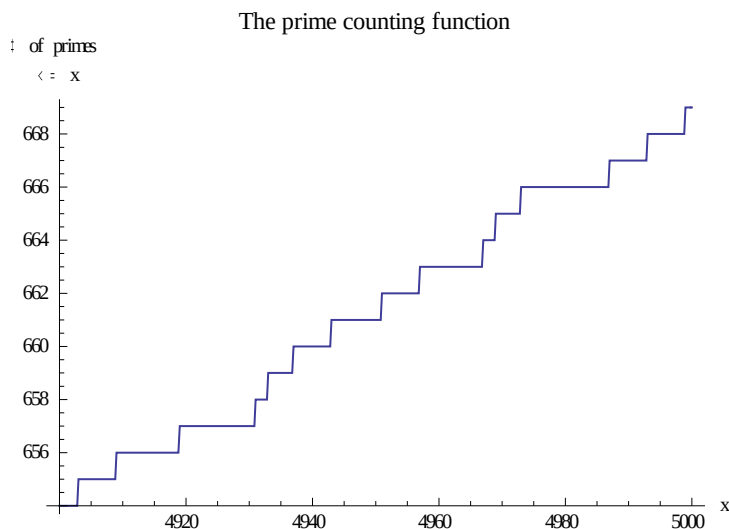
Observations?



Observations?



It may have appeared to 'smooth out', but upon blowing up, still has the jagged appearance:



Gauss (at a young age) guessed the overall rough outline of the 'curve' [he actually guessed the slope, as $1/\ln(x)$]

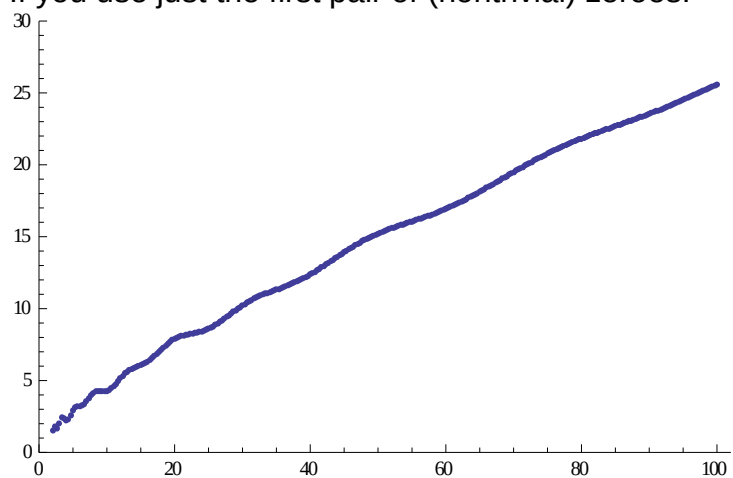
Test it out:

The number of primes between 100,000 and 101,000 is ..? 100,003 is prime... 100,999 is prime; altogether, 81 numbers in that range are prime.

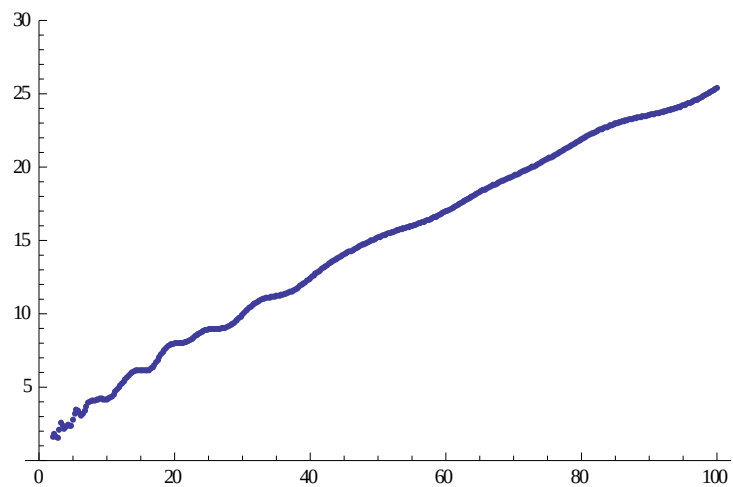
And $1/\ln(100,000)$ is $1/11.5$, i.e. we expect about $1/11.5$ numbers to be prime, or $1000/11.5$, or $86.85\dots$, i.e we'd expect 87 primes and there are 81. Not bad.

Riemann (Gauss's student!) figured out a way to improve, and get the fine detail structure, using the zeroes of the zeta function. (The nontrivial ones, really.)

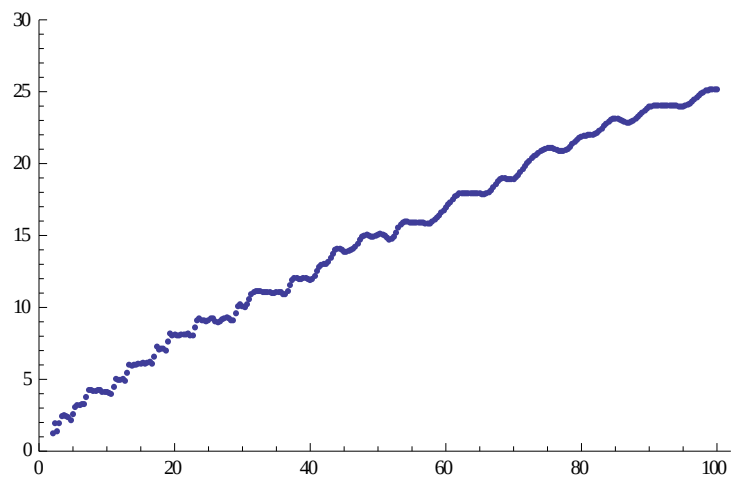
If you use just the first pair of (nontrivial) zeroes:



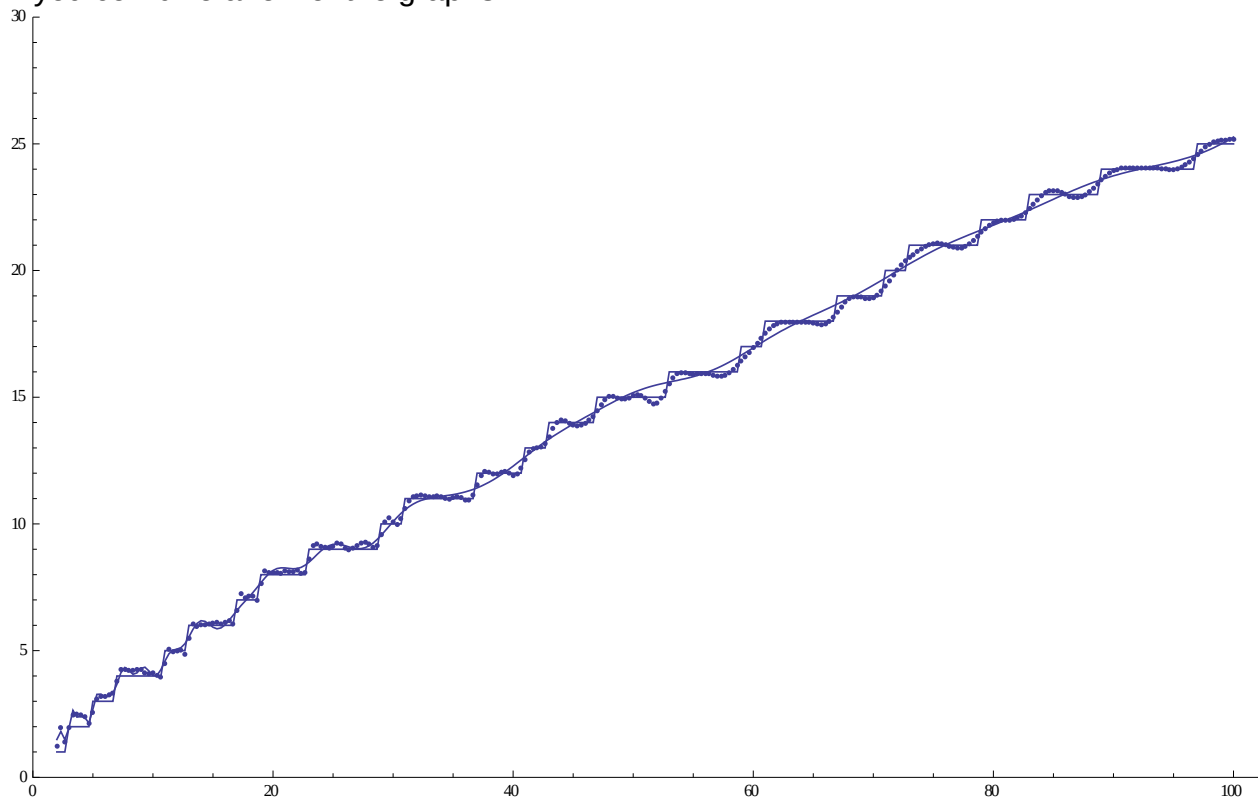
If you use the first 3 pairs of zeroes:



If you use the first 45 zeroes:

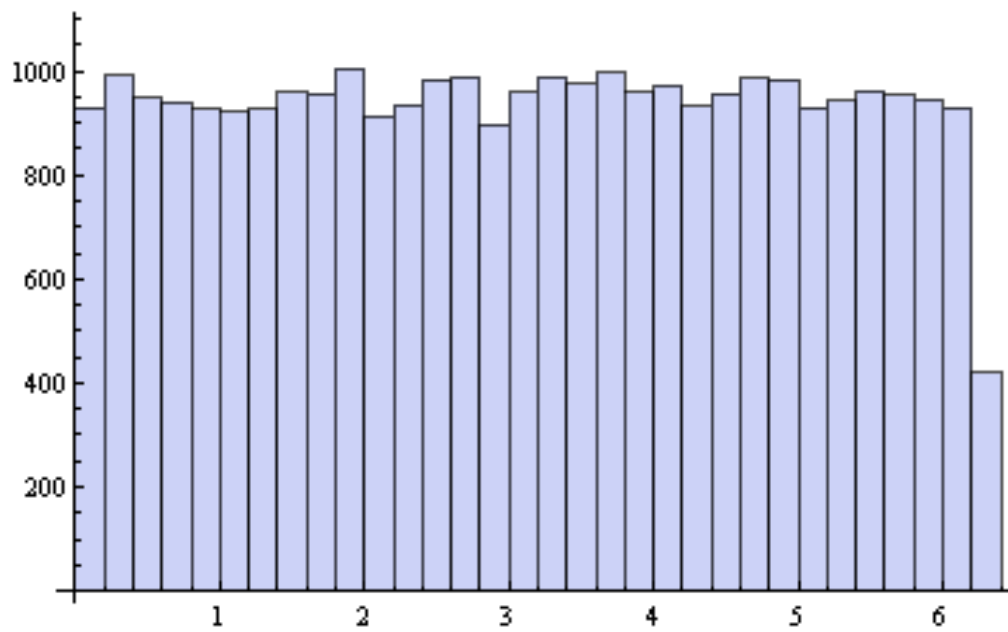


If you combine a few of the graphs:

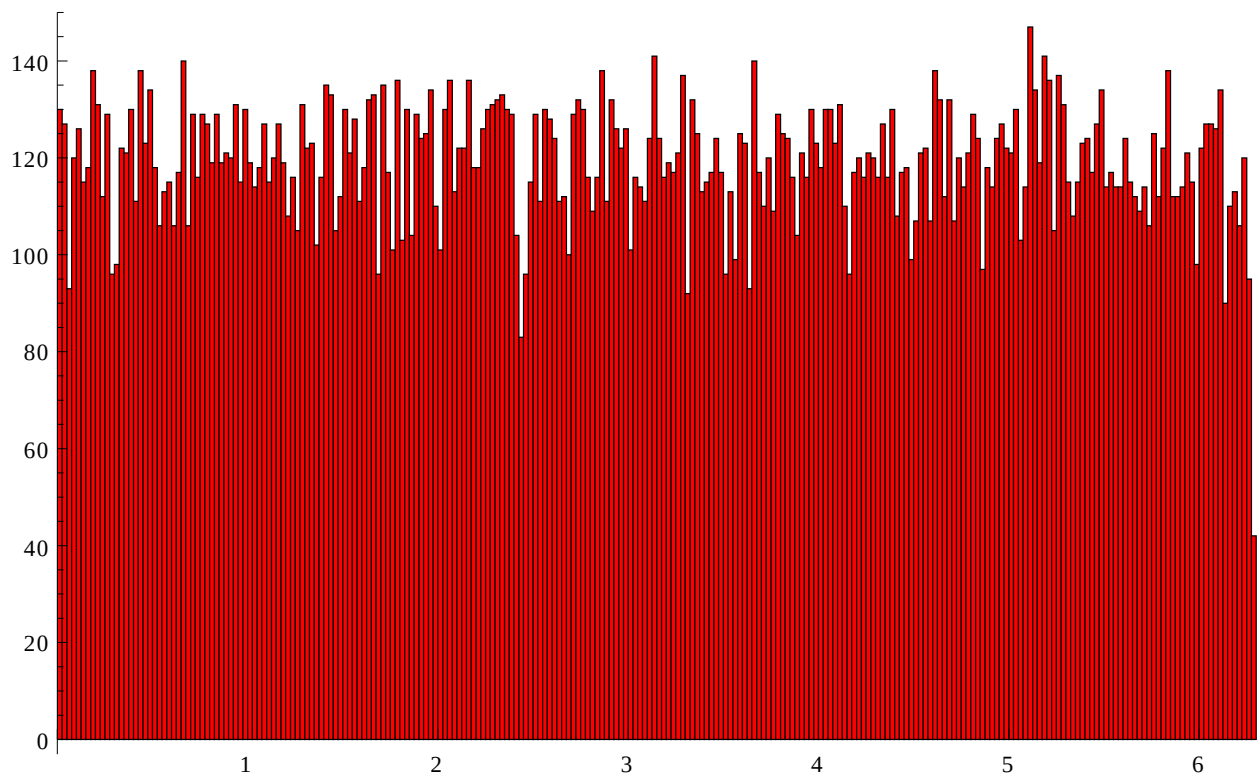


The MOST FAMOUS/MOST IMPORTANT problem in number theory, perhaps all of mathematics: The Riemann Hypothesis. We know the zeroes of Riemann's zeta function tell us about the primes.

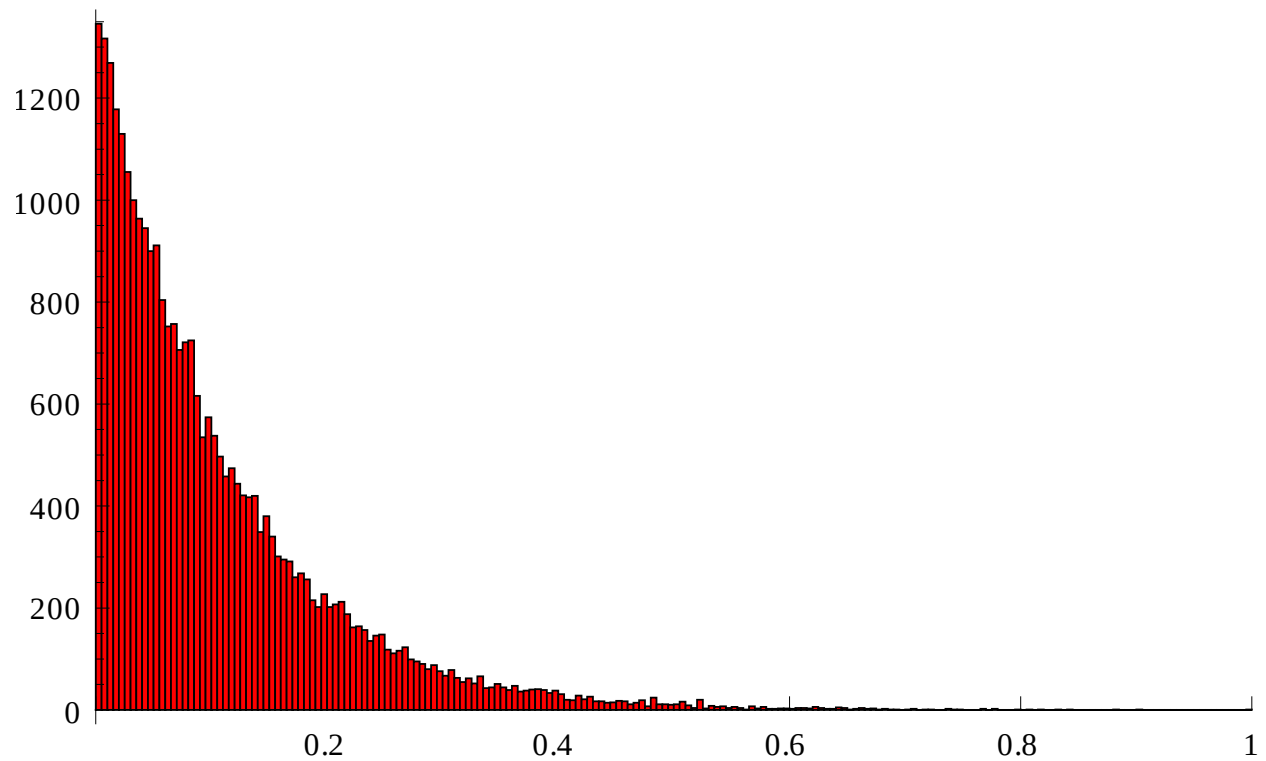
But there's also perhaps a deep connection, it seems, with physics.



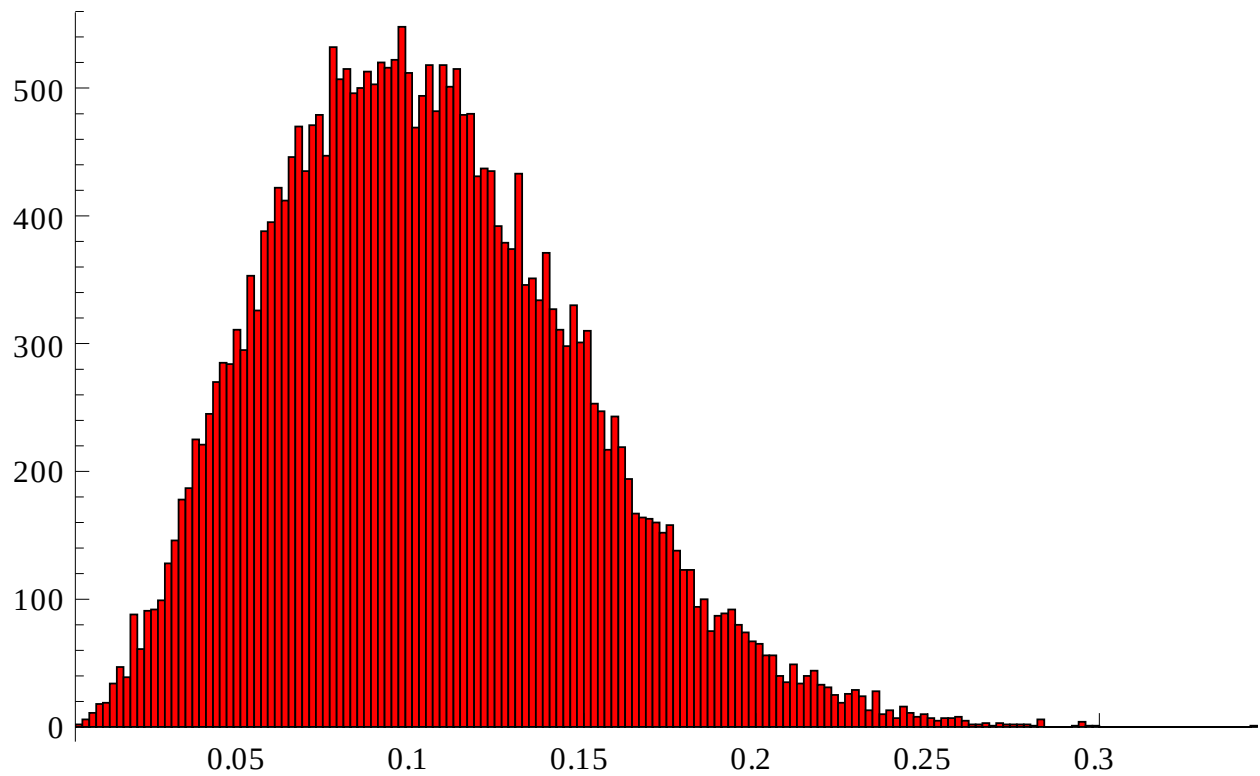
Histogram (distribution) of 500 sets of 60 #s uniformly at random between 0 and 2π



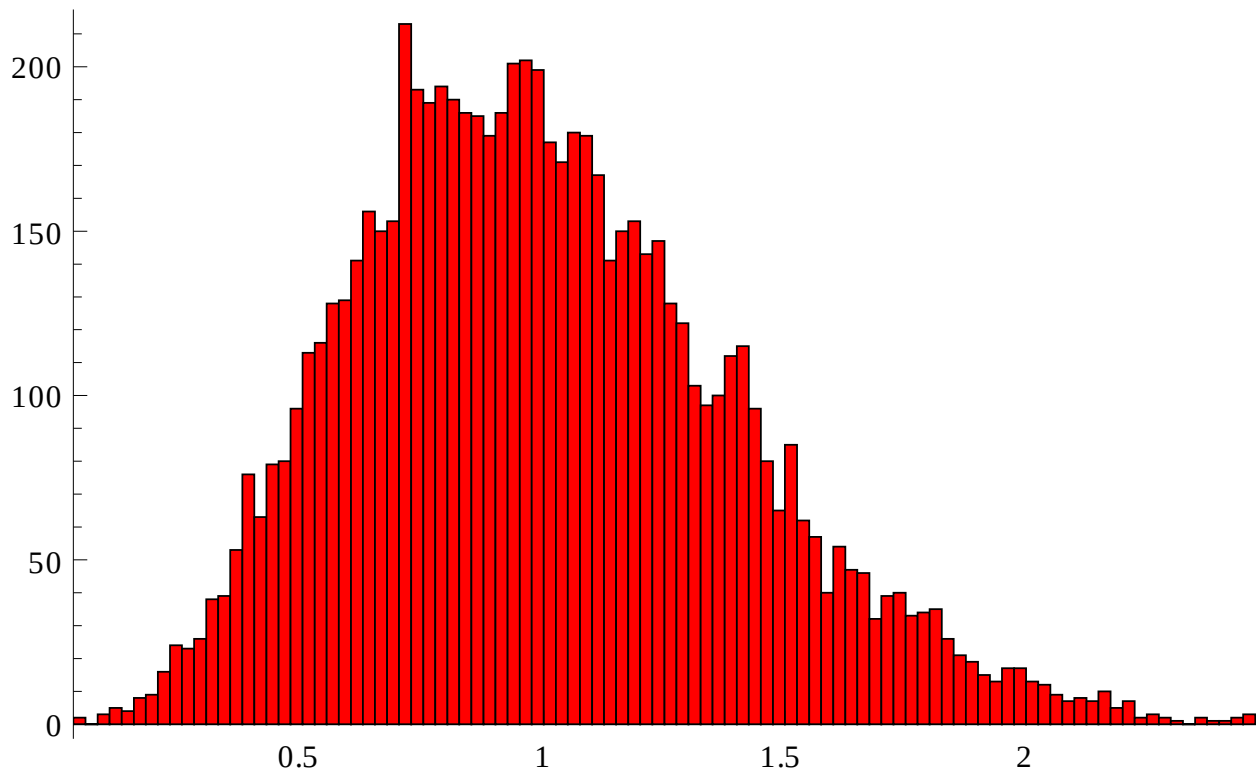
Histogram (distribution) of 500 sets of 60 #s uniformly at random between 0 and 2π
(just more finely dividing up the 30,000 #s)



Distribution of the **spacing** between those 60 random #s uniformly at random between 0 and 2π ; looks like a typical exponential distribution. Average spacing is intuitively about $2\pi/60$ or about .1



Histogram (i.e. distribution) of **spacing** between the (complex argument of the) eigenvalues of 500 random 60x60 unitary matrices. Intuitively, average spacing is $2\pi/60$ or about .1. Does NOT look exponential; looks like there's 'repulsion'



Histogram (distribution) of the (normalized) **spacing** between the first 7500 nontrivial zeroes of the Riemann zeta function; zeroes are normalized so that average spacing is 1. Does not look exponential; again, looks like there's 'repulsion'.

Remember that Zeta(s), initially defined for $\text{real}(s) > 1$ as $\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$, can be analytically continued into the complex s plane, and has a simple pole at $s=1$. Actually, $\zeta(s) = 1/(s-1) + c_0 + c_1(s-1) + c_2(s-1)^2 + c_3(s-1)^3 + \dots$ - think that zeta is $1/(s-1) +$ a nice function analytic everywhere with a Taylor series. The constant term, c_0 , is Euler's gamma constant, i.e. $c_0 = \gamma = .57721566\dots$ and all the coefficients $c_0, c_1, c_2, \dots$ are essentially called generalized Euler constants or Stieltjes constants. Do a google search for Stieltjes constants if interested.