

MATH 421 MIDTERM II, TAKE-HOME SOLUTIONS, FALL TERM 2009

1. Let $A \subset \mathbb{R}$ have the property that $f(A)$ is bounded for every continuous function $f : A \rightarrow \mathbb{R}$. Prove that A is compact. [This is something of a converse of the Extreme Value Theorem.]

Proof: Consider the function $f(x) = x$. It is continuous on all of $\mathbb{R}$, so it is certainly continuous on A . But $f(A) = A$, and so by the problem hypothesis we can conclude A is bounded.

Now consider a sequence $(a_n)_{n=1}^\infty$ of points $a_n \in A$ which converges to some $a \in \mathbb{R}$ – but assume that $a \notin A$. Define a function f by $f(x) = \frac{1}{|x-a|}$ on $\mathbb{R} \setminus \{a\}$. This f is continuous on its domain, by Theorem 4.3.4(iv). Since we are assuming $a \notin A$, it follows that f is continuous on A and so $f(A)$ is bounded. Therefore we may pick M such that $|x| < M \ \forall x \in f(A)$, i.e., $|f(b)| < M \ \forall b \in A$.

Since $a_n \rightarrow a$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < 1/M \ \forall n \geq N$. In particular, if $n = N + 1$, then $|a_n - a| < 1/M$ and hence $f(a_n) = \frac{1}{|a_n - a|} > M$ – but this contradicts the definition of M . Therefore our assumption that $a \notin A$ must be false. This means A contains all of its limit points and is therefore closed.

Since A is closed and bounded it is compact, by the Heine-Borel Theorem. $\square$

2. Let $D \subset \mathbb{R}$ be a bounded set and $f : \mathbb{R} \rightarrow \mathbb{R}$ a uniformly continuous function. Prove that $f(D)$ is bounded as well. Is the same true if f is merely continuous (not *uniformly* continuous)? Prove your answer, if it is “yes”, or provide a counter-example if it is “no”.

Proof: D is bounded, so its closure $\overline{D}$ is, too (by Theorem 3.2.12 the closure is the intersection of all closed sets containing D , and there are certainly bounded closed sets containing D if D is bounded – like $[-M, M]$ for some $M > 0$). But therefore $\overline{D}$ is compact, by Heine-Borel, and by the preservation of compact sets under continuous functions (Theorem 4.4.2), $f(\overline{D})$ is compact as well. In particular, $f(\overline{D})$ is bounded (H-B again), and so $f(D)$ is a subset of the bounded set $f(\overline{D})$ and thus is bounded itself.

This proof did not use uniform continuity (the hypothesis of Theorem 4.4.2 is just continuity), so the answer to the second part of the question is “yes, we already proved it.”

Proof of the more interesting version with “ $f : A \rightarrow \mathbb{R}$ is uniformly continuous...” D is bounded, say $|d| < M \ \forall d \in D$. Also, f is uniformly continuous, so $\exists \delta > 0$ s.t. $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Now choose an $N \in \mathbb{N}$ large enough so that $2M/N < \delta$. Divide the interval $[-M, M]$ into N equal closed subintervals – so each of size $2M/N$ – and call them $I_1, \dots, I_N$. For each of these subintervals, such as I_j , if $I_j \cap D \neq \emptyset$, define d_j to be any point you like in $I_j \cap D$. Finally, define

$$\mathcal{M} = 1 + \max \{ |f(d_j)| \mid j \in \{1, \dots, N\} \text{ s.t. } d_j \text{ is defined} \} .$$

We claim that this $\mathcal{M}$ is a bound for $f(D)$, i.e., $|f(d)| < \mathcal{M} \ \forall d \in D$. To see this, take any $d \in D$. It is in some I_j , in fact in one of the I_j for which the corresponding d_j is defined. Furthermore, since the size of each I_j is $2M/N$ and $2M/N < \delta$, we see that $|d - d_j| < \delta$ as well. But then by the definition of δ , we have that (first part by the triangle inequality)

$$|f(d)| - |f(d_j)| \leq |f(d) - f(d_j)| < 1$$

and so

$$|f(d)| < |f(d_j)| + 1 \leq \mathcal{M}$$

as desired. $\square$

Counterexample if $f : A \rightarrow \mathbb{R}$ is only continuous: Let $D = [-1, 0) \cup (0, 1]$ and $f(x) = 1/x$. 0 is a limit point of D , and so by exactly the argument used in the solution of problem 1, above, f is not bounded on D – or, said another way, the set $f(D)$ is not bounded. But this f is continuous on its domain (by Theorem 4.3.4(iv) again, if you like), but not uniformly continuous there (by a homework exercise we have done). This counterexample shows that we needed uniform continuity in this exercise. $\square$

3. [The following is sometimes called the “Contraction Mapping Theorem”; it is also (part of) exercise 4.3.9 from the book.] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following: there exists a constant c such that $0 < c < 1$ and for which

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

a) Show that $f(x)$ is continuous on $\mathbb{R}$.

Proof: Let $c \in \mathbb{R}$. Given $\epsilon > 0$, let $\delta = \epsilon/c$. Now if $|x - c| < \delta$, it follows that

$$|f(x) - f(y)| \leq c|x - y| < c\delta = \epsilon$$

and hence f is continuous by Theorem 4.3.2(i). $\square$

b) Pick any $y_1 \in \mathbb{R}$ and then inductively define $y_{n+1} = f(y_n)$ for $n \geq 1$. Prove that the sequence $(y_n)_{n=1}^\infty$ is Cauchy. Hence we can define $y = \lim_{n \rightarrow \infty} y_n$.

Proof: Let $a = |y_1 - y_2|$. Note that

$$\begin{aligned} |y_2 - y_3| &= |f(y_1) - f(y_2)| \leq c|y_1 - y_2| = ac \\ |y_3 - y_4| &= |f(y_2) - f(y_3)| \leq c|y_2 - y_3| \leq ac^2 \\ &\vdots \\ |y_n - y_{n+1}| &= |f(y_{n-1}) - f(y_n)| \leq c|y_{n-1} - y_n| \leq ac^{n-1} \end{aligned}$$

That is, the distances between successive y_n 's follow a geometric sequence, and this gives us the Cauchy property since the tails of geometric series get as small as one likes by going far enough down the tail. In more detail: if $m > n$,

$$\begin{aligned} |y_n - y_m| &= |y_n - y_{n+1} + y_{n+1} - y_{n+2} + \cdots - y_m| \quad \text{adding and subtracting like terms} \\ &\leq |y_n - y_{n+1}| + \cdots + |y_{m-1} - y_m| \quad \text{triangle inequality} \\ &\leq ac^{n-1} + \cdots + ac^{m-2} \quad \text{from above} \\ &= \frac{ac^{n-1} - ac^{m-1}}{1 - c} \quad \text{sum of finite geometric series} \\ &< \frac{ac^{n-1}}{1 - c} \quad \text{since } ac^{m-1} \text{ is positive} \end{aligned}$$

Now, given $\epsilon > 0$, let $N \in \mathbb{N}$ be greater than $1 + \ln(\epsilon(1 - c))/\ln(c)$, so that $\frac{ac^{n-1}}{1 - c} < \epsilon$ whenever $n \geq N$ (check the algebra!). Hence, $|y_n - y_m| < \epsilon$ whenever $n, m \geq N$ and so the sequence is Cauchy. $\square$

Cauchy sequences are convergent, so $\lim_{n \rightarrow \infty} y_n$ exists, we can call it y .

c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and in fact it is the only fixed point.

Proof: Suppose not, so that $f(y) \neq y$. The idea here is that we can go far enough out in the sequence such that the elements are both very close to y , and yet that f moves those sequence elements very little. It's an $\epsilon/3$ argument, using the triangle inequality, breaking the step between y and $f(y)$ into smaller steps $y \mapsto y_n \mapsto f(y_n) = y_{n+1} \mapsto f(y)$. Details follow:

Define $\Delta = |f(y) - y|$. Since $y_n \rightarrow y$, we can choose an $N_1 \in \mathbb{N}$ such that $|y - y_n| < \Delta/3$ whenever $n \geq N_1$.

We have seen that $(y_n)_{n=1}^\infty$ is Cauchy, so there exists $N_2 \in \mathbb{N}$ such that $|y_n - y_m| < \Delta/3$ when $n, m \geq N_2$.

Letting $N = \max\{N_1, N_2\}$, if $n \geq N$ we have

$$\begin{aligned} \Delta &= |f(y) - y| = |f(y) - f(y_{n+1}) + f(y_{n+1}) - y_{n+1} + y_{n+1} - y| \quad \text{adding and subtracting like terms} \\ &\leq |f(y) - f(y_{n+1})| + |f(y_{n+1}) - y_{n+1}| + |y_{n+1} - y| \quad \text{triangle inequality} \\ &\leq c|y - y_{n+1}| + |y_{n+2} - y_{n+1}| + |y_{n+1} - y| \quad \text{Lipschitz property and definition of } y_n \\ &\leq c \cdot \Delta/3 + \Delta/3 + \Delta/3 \quad \text{definition of } N \\ &< \Delta \quad \text{since } c < 1 \end{aligned}$$

and this contradiction shows that our assumption that $f(y) \neq y$ must be incorrect. $\square$ The uniqueness is even easier: f moves pairs of points *closer together* (by a factor of c), which is impossible if we have two points which are not moved at all:

Proof that the fixed point is unique: Say f has two fixed points x and y , so $f(x) = x$ and $f(y) = y$. Then

$$|x - y| = |f(x) - f(y)| \leq c|x - y|$$

which is impossible unless $|x - y| = 0$, since $c < 1$. Thus $x = y$ and the fixed point of f is unique. $\square$

4. [The following is exercise 4.3.10 from the book.] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following:

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

- a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Proof: Just use the given algebraic identity: $f(17) = f(17 + 0) = f(17) + f(0)$, so subtracting $f(17)$ from both sides gives that $0 = f(0)$. Then, for any x , $0 = f(0) = f(x - x) = f(x) + f(-x)$ so $f(-x) = -f(x)$. $\square$

- b) Show that if f is continuous at 0, then it is continuous on all of $\mathbb{R}$.

Proof: Given any $c \in \mathbb{R}$, we wish to show that $f(x)$ is continuous at c . But

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow 0} f(x + c) = \lim_{x \rightarrow 0} (f(x) + f(c)) = \lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} f(c) = \left(\lim_{x \rightarrow 0} f(x) \right) + f(c) \\ &= f(0) + f(c) = f(c) \end{aligned}$$

so we have the desired continuity at c , by Theorem 4.3.2 (if you like). $\square$

- c) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$ and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for all $r \in \mathbb{Q}$.

Proof: For $n \in \mathbb{N}$, $f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = kn$, where these sums are each of n terms. If $z \in \mathbb{Z}$ is negative, then $-z \in \mathbb{N}$ so by what we have just proven and part a, $f(z) = -f(-z) = -(k(-z)) = kz$. Now, if $r = p/q \in \mathbb{Q}$, $qr \in \mathbb{Z}$, so $qf(r) = f(r) + \cdots + f(r) = f(r + \cdots + r) = f(qr) = kqr$, from which $f(r) = kr$, also. $\square$

- d) Prove that if f is continuous at 0 then in fact $f(x) = kx$ for all $x \in \mathbb{R}$ (k as in part c)) – that is, f must necessarily be a linear function through the origin.

Proof: We know $f(r) = kr \forall r \in \mathbb{Q}$. If we instead take an arbitrary $x \in \mathbb{R}$, we can find a sequence $(r_n)_{n=1}^\infty$ of rational numbers $r_n \in \mathbb{Q}$ for which $r_n \rightarrow x$. Note also that by the hypothesis of this part plus the result of the last part, we can use the continuity of f at x . Therefore $f(x) = \lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} kr_n = k \lim_{n \rightarrow \infty} r_n = kx$. $\square$

XC) Can you find an example of a function which satisfies the initial additive condition above ($f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$) but is *not* a linear function through the origin?

No proof or example.... There are examples, but they are not easy to describe. By part b, this amounts to constructing an additive function on $\mathbb{R}$ which is not even continuous at 0, which is possible but very intricate. The only way I know to do this requires tools we have not discussed in this class.

5. [The following is exercise 5.3.7 from the book.] Recall that we say a function $f : (a, b) \rightarrow \mathbb{R}$ is *nondecreasing* on (a, b) if $f(x) \leq f(y)$ whenever $x, y \in (a, b)$ satisfy $x \leq y$.

- a) Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Prove that f is nondecreasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Proof: Assume that a differentiable function $f : (a, b) \rightarrow \mathbb{R}$ is nondecreasing on its domain. Pick any point $x \in (a, b)$ and consider $f'(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}$. If $y > x$ then by the nondecreasing hypothesis, the difference quotient is a non-negative number divided by a positive number; if $y < x$ it is a non-positive number divided by a negative number. Hence the limit is of only non-negative numbers and thus $f'(x) \geq 0$.

Conversely, assume that a differentiable function $f : (a, b) \rightarrow \mathbb{R}$ satisfies $f'(z) \geq 0$ for all $z \in (a, b)$. Given any two $x, y \in (a, b)$ for which $y > x$, the Mean Value Theorem tells us that $f(y) - f(x) = (y - x) \cdot f'(z)$ for some $z \in (x, y)$. But by our hypotheses about x and y and about the sign of f' , this gives $f(y) - f(x)$ as the product of a positive number and a non-negative one. Hence f is non-decreasing. $\square$

- b) Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Prove, however, that g is *not* nondecreasing on any open interval containing 0.

Proof: The book shows in §5.1 that $g_2(x) = g(x) - x/2$ is differentiable everywhere and has $g'_2(0) = 0$. Therefore $g(x)$ is also differentiable and $g'(0) = 1/2 + g'_2(0) = 1/2 > 0$, by Theorem 5.2.4(i).

Consider now any open interval I containing 0. Think about the part of that interval away from 0: there we can compute $g'(x) = 1/2 + 2x \sin(1/x) - \cos(1/x)$. If both $|x| < 1/8$ – so that $|2x \sin(1/x)| < 1/4$ – and $x \in (\frac{1}{\cos^{-1}(.75)+2\pi n}, \frac{1}{-\cos^{-1}(.75)+2\pi n})$ – so that $\cos(1/x) > .75$ – we have $g'(x) < 0$. Note that as $n \rightarrow \infty$, these intervals $J_n = (\frac{1}{\cos^{-1}(.75)+2\pi n}, \frac{1}{-\cos^{-1}(.75)+2\pi n})$ get closer and closer to 0, and therefore for sufficiently large n , J_n will be contained in our interval I , and will consist entirely of numbers x for which $|x| < 1/8$. Therefore on such J_n , g' will be strictly negative and hence by part **a**, g will not be nondecreasing, finishing the problem. $\square$