

MATH 421 MIDTERM II
TAKE-HOME PART
FALL TERM 2009

INSTRUCTIONS:

- This is an *open textbook and notes* test, so feel free to use the course textbook and your class notes. However, **you must work on your own**. Also, **you may not consult other books or the Internet**.
- If you are getting stuck on a problem, feel free to contact me and I will (almost always) give you a hint.
- Exposition is **very important**: please be as clear as you can about what you are doing. Be organized. Make a rough draft, then put your more organized thoughts into a final draft.... You get the idea.
- This is **due in class on Tuesday, November 10th**.

DO ANY THREE OF THE FOLLOWING PROBLEMS:

1. Let $A \subset \mathbb{R}$ have the property that $f(A)$ is bounded for every continuous function $f : A \rightarrow \mathbb{R}$. Prove that A is compact. [*This is something of a converse of the Extreme Value Theorem.*]
2. Let $D \subset \mathbb{R}$ be a bounded set and $f : \mathbb{R} \rightarrow \mathbb{R}$ a uniformly continuous function. Prove that $f(D)$ is bounded as well.
Is the same true if f is merely continuous (not *uniformly* continuous)? Prove your answer, if it is “yes”, or provide a counter-example if it is “no”.
3. [*The following is sometimes called the “Contraction Mapping Theorem”; it is also (part of) exercise 4.3.9 from the book.*] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following: there exists a constant c such that $0 < c < 1$ and for which

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

- a) Show that $f(x)$ is continuous on $\mathbb{R}$.
 - b) Pick any $y_1 \in \mathbb{R}$ and then inductively define $y_{n+1} = f(y_n)$ for $n \geq 1$. Prove that the sequence $(y_n)_{n=1}^{\infty}$ is Cauchy. Hence we can define $y = \lim_{n \rightarrow \infty} y_n$.
 - c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and in fact it is the only fixed point.
4. [*The following is exercise 4.3.10 from the book.*] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following:

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

- a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
 - b) Show that if f is continuous at 0, then it is continuous on all of $\mathbb{R}$.
 - c) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$ and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for all $r \in \mathbb{Q}$.
 - d) Prove that if f is continuous at 0 then in fact $f(x) = kx$ for all $x \in \mathbb{R}$ (k as in part c)) – that is, f must necessarily be a linear function through the origin.
- XC) Can you find an example of a function which satisfies the initial additive condition above ($f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$) but is *not* a linear function through the origin?
5. [*The following is exercise 5.3.7 from the book.*] Recall that we say a function $f : (a, b) \rightarrow \mathbb{R}$ is *nondecreasing* on (a, b) if $f(x) \leq f(y)$ whenever $x, y \in (a, b)$ satisfy $x \leq y$.
- a) Assume $f : (a, b) \rightarrow \mathbb{R}$ is differentiable on (a, b) . Prove that f is nondecreasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.
 - b) Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Prove, however, that g is *not* nondecreasing on any open interval containing 0.