

MATH 421 MIDTERM I
TAKE-HOME PART
FALL TERM 2009
SOLUTIONS

1. Assume that (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Prove that the entire sequence (a_n) must converge to this a .
[This is Exercise 2.5.4. from the book.]

[We'll think out loud for this one, while constructing a complete proof.]

The problem here seems to be to get the whole thing to converge – once it does, since it is a subsequence of itself, it will converge to a . And many subsequences do converge, but there are **many** subsequences of a given sequence, in fact uncountably many.

[Why? Well, let the set of subsequences of a given sequence be $\mathcal{S}$, and let $\mathcal{P}$ be the set of infinite subsets of $\mathbb{N}$. Note that $\mathcal{P}$ is uncountable.

[[Why? Well, the power set $P(\mathbb{N})$ is uncountable by Cantor's Theorem and the set $\mathcal{F}$ of *finite* subsets of $\mathbb{N}$ is countable by a previous homework, and if $\mathcal{P}$ were countable, we would then have an expression of the uncountable set $P(\mathbb{N})$ as the union $\mathcal{F} \cup \mathcal{P}$ of two countable sets; therefore $\mathcal{P}$ is uncountable]]

Now notice that a subsequence of a given sequence amounts to a choice of some of the elements of the original sequence as “in” and the rest as “out” ... but the “in” ones keep going out to infinity, they never stop (so that in the end they form a complete sequence themselves). Thus a subsequence corresponds exactly to an element of $\mathcal{P}$. Hence $\mathcal{S}$ is bijective with $\mathcal{P}$ and so uncountable.]

But for any particular element of the sequence, we can add it on to *any* subsequence without changing whether that sequence converges or not [convergence depends only upon the tails of sequences, so is completely unaffected by messing around with a finite bit at the beginning]... the only problem is if we try to add infinitely many elements to a convergent subsequence it may no longer be convergent – to a , of course, by the hypothesis.

Hmm, but if it is no longer convergent, that means these infinitely many extra elements are not tending to a , so infinitely many of them are “too far” away from a . How would this contradict the give hypothesis? Maybe from these infinitely many “bad” (too far from a) elements, we could construct something that contradicted that hypothesis...

Since the hypothesis only gives information when we have a convergent subsequence (that information being that the subsequence then converges to a), we would need to find a convergent subsequence of the bad elements. How do we automatically extract convergent subsequences of given sequences? Well, this smells like the Bolzano-Weierstrass Theorem, which certainly applies here because our sequence was given to us as bounded (so the bad elements are also bounded).

This seems to be suggesting a strategy: proof that the whole sequence converges by contradiction; take the infinitely many elements which are not tending towards a and extract a convergent subsequence by B-W and apply the hypothesis. Let's do it.

Assume the entire sequence does not converge to a . By the definition of convergence, this means: $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N} \exists n \geq N$ with $|a_n - a| > \epsilon$. Let $\epsilon > 0$ be as in this statement and first set $N = 1$. Then there is an $n_1 \geq N$ such that $|a_{n_1} - a| > \epsilon$. Now for all $k \in \mathbb{N}$, $k > 1$, let n_k be the index greater than n_{k-1} (so use $N = n_{k-1} + 1$ in the statement of non-convergence) for which $|a_{n_k} - a| > \epsilon$. That is, we have used the non-convergence to find a subsequence of the original sequence which is entirely more than ϵ away from a . But now this subsequence is itself a bounded sequence (it's a subsequence of a bounded sequence) and so has in turn a sub-subsequence $(a_{n_{k_j}})_{j=1}^{\infty}$ which converges. This $(a_{n_{k_j}})_{j=1}^{\infty}$ is in fact also a subsequence of $(a_n)_{n=1}^{\infty}$ so by our hypothesis, being convergent it converges to a . This is, however, impossible, since every element of $(a_{n_{k_j}})_{j=1}^{\infty}$ is at least our fixed ϵ away from a – in fact, the sequence $(a_n)_{n=1}^{\infty}$ by exactly the definition of non-convergence to a .

This contradiction tells us that the whole sequence does converge to a . $\square$

2. Let (a_n) be a bounded sequence and define a set

$$S = \{x \in \mathbb{R} \mid x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) of which converges to the number $s = \sup S$. [This is a direct approach to the Bolzano-Weierstrass Theorem – so don't use B-W in your proof! Note also that this is Exercise 2.5.6 from the book.]

[Let's follow the hint of taking the direct approach. We'll also keep in mind the general trend that when using the supremum, both parts of its definition – it's an **upper bound**, and it's the **least such** – almost always are used.]

Fix an $k \in \mathbb{N}$. Since $s = \sup S$ and $s - 1/k < s$, s cannot be an upper bound of S by the definition of the supremum (the supremum is the *least* upper bound, so anything strictly less than the supremum cannot be an upper bound at all). Thus there is an $x \in S$ for which $x > s - 1/k$. Hence, by the definition of S , there are infinitely many a_n greater than this x , and hence also greater than $s - 1/k$.

Also by the definition of the supremum (it's an upper bound), anything strictly greater than the supremum is strictly greater than all of the elements of the set. Thus $s + 1/k$ is greater than all of S , and in particular it is not in S . Hence, by the definition of S , there are only finitely many a_n greater than $s + 1/k$.

We have learned therefore that there are infinitely many a_n greater than $s - 1/k$ and finitely many greater than $s + 1/k$. There will therefore necessarily be able to find an index n_k which is greater than all of the n 's for which $a_n \geq s + 1/k$ but still one of the ones for which $a_{n_k} > s - 1/k$. That is, this index n_k will make $|a_{n_k} - s| < 1/k$. In addition, if $k > 1$, we can also choose this n_k to satisfy $n_k > n_{k-1}$.

We have therefore defined a subsequence $(a_{n_k})_{k=1}^{\infty}$ satisfying $|a_{n_k} - s| < 1/k$. It is easy to show that this subsequence converges to s : given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $N > 1/\epsilon$ (so $1/N < \epsilon$). It follows that for $k \geq N$, $|a_{n_k} - s| < 1/k \leq 1/N < \epsilon$. Thus $\lim_{k \rightarrow \infty} a_{n_k} = s$. $\square$

3. Decide whether each of the following sets is countable or uncountable, and prove your claim:

- The set of **algebraic numbers**, that is, numbers $x \in \mathbb{R}$ which are roots of polynomial equations $a_n x^n + \dots + a_1 x + a_0 = 0$ with integer coefficients, (so $a_0, a_1, \dots, a_n \in \mathbb{Z}$). [Hint: think about first whether the set of such equations is countable or not, and then you may use the basic algebraic result that every polynomial of degree n has at most n real roots. Note also that this is (part of) Exercise 1.4.12 from the book.]
- The set of functions from the unit interval $[0, 1] \subset \mathbb{R}$ to the rationals, $\{f : [0, 1] \rightarrow \mathbb{Q}\}$.
- EXTRA CREDIT:** What about the set of *continuous* functions on the unit interval with values in the rationals, $\{f : [0, 1] \rightarrow \mathbb{Q} \mid f \text{ is continuous}\}$? [Hint: what must be true of all such functions? Think about the Intermediate Value Theorem from calc I and Corollary 1.4.4 from the book.] What about the set of continuous real-valued functions, $\{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$?

[Again doing a lot of thinking out loud before making a formal proof.]

a) We are allowed to use the fact that there are only finitely many roots for any particular polynomial. Since adding up a finite number of things (even a countable number of times) doesn't make countable things uncountable, we really need only figure out how many integer-coefficient polynomials there are. Now for a particular degree n , there are n coefficients, and there are countably many choices for each of these coefficients – so again we seem to be in a situation of a finite sum (with n terms) of countable things... so it looks like there should be countable number of algebraic numbers. Let's do this more formally.

Let $\mathcal{A}$ be the set of algebraic numbers. The usual notation for the set of polynomials with integer coefficients is $\mathbb{Z}[x]$. Then by the above definition, if we write $\mathcal{R}_p$ for the (finite!) set of roots of a particular $p \in \mathbb{Z}[x]$, we have

$$\mathcal{A} = \{\text{all roots of } p \mid p \in \mathbb{Z}[x]\} = \bigcup_{\substack{p \in \mathbb{Z}[x] \\ \deg p > 0}} \mathcal{R}_p$$

(note we can restrict to polynomials of non-zero degree because degree-zero polynomials have no roots). By the “countable unions of countable sets are countable” (and, even easier, “finite unions of countable sets are

countable” and “countable unions of finite sets are countable”) theorem, it therefore suffices to show that $\mathbb{Z}[x]$ is countable.

Since every polynomial has some degree, we can write $\mathbb{Z}[x] = \cup_{d \in \mathbb{N}} \mathbb{Z}_d[x]$, where we use the notation $\mathbb{Z}_d[x] = \{p \in \mathbb{Z}[x] \mid \deg p = d\}$ for the polynomials of degree d . Again by the theorem about countable unions, it suffices to show that each $\mathbb{Z}_d[x]$ is countable.

A polynomial of degree d is determined by its d coefficients and, conversely, any d coefficients – starting with a non-zero coefficient, so the degree will indeed be d – determines such a polynomial. Thus we have a bijection between $\mathbb{Z}_d[x]$ and the set $C^d = \{(a_d, a_{d-1}, \dots, a_1, a_0 \in \mathbb{Z}^{d+1} \mid a_d \neq 0\}$ (recall $\mathbb{Z}^n$ denotes the set of n -tuples of integers). It therefore suffices to show that C^d is countable.

Note that C^d is certainly infinite if $d > 1$ – the last coefficient a_0 can be anything. Also, $C^d \subset \mathbb{Z}^{d+1}$, so it suffices to show that $\mathbb{Z}^n$ is countable for any $n \in \mathbb{N}$, because a countable set cannot have an uncountable subset (by a theorem from the book and class).

Finally, we prove that $\mathbb{Z}^n$ is countable for all $n \in \mathbb{N}$ by induction on n . The initial case is to show that $\mathbb{Z}^1$, which is nothing other than $\mathbb{Z}$, is countable. But $\mathbb{Z}$ is the union of two copies of $\mathbb{N}$ (for the positive and negative numbers and a singleton set $\{0\}$), so is clearly countable. Now assume that $\mathbb{Z}^n$ is countable and consider $\mathbb{Z}^{n+1}$. We can write $\mathbb{Z}^{n+1} = \cup_{a \in \mathbb{Z}} \mathbb{Z}_a^{n+1}$, where $\mathbb{Z}_a^{n+1} = \{(a, b_1, b_2, \dots, b_n) \in \mathbb{Z}^{n+1}\}$ is the collection of $(n+1)$ -tuples in $\mathbb{Z}^{n+1}$ whose first element is the integer a . But there is a simple bijection between $\mathbb{Z}_a^{n+1}$ and $\mathbb{Z}^n$ (defined as “drop the first element”), hence $\mathbb{Z}_a^{n+1}$ is countable. Therefore $\mathbb{Z}^{n+1}$ is a countable union of countable sets and is itself countable. $\square$

[This is the “start at the conclusion, walk backwards until you bump into the hypotheses” strategy of proof.]

b) There are a lot of functions – for example, there is the set of constant functions, which set is bijective with $\mathbb{Q}$, so it looks like this set of functions is quite big, probably strictly bigger than $\mathbb{Q}$ itself. Hence we

Claim: The set $\mathcal{F} = \{f : [0, 1] \rightarrow \mathbb{Q}\}$ is uncountable.

Proof: Let’s describe a bijection between this set of functions and some set which we know to be uncountable. For example, the power set of the unit interval, $P([0, 1])$ (as discussed in the book, $[0, 1]$ is of the same cardinality as $\mathbb{R}$ itself, and Cantor’s Theorem tells us that $P([0, 1])$ is of a strictly larger cardinality).

Actually, it will suffice to have a 1-1 map from $P([0, 1])$ into $\mathcal{F}$. The range of this function will then be bijective with $P([0, 1])$ (functions are always surjective onto their images, and we are going to prove our map is 1-1), and so this range will be an uncountable subset of $\mathcal{F}$, which, as we have seen several times, means $\mathcal{F}$ itself must be uncountable (since countable sets can only have empty, finite or countable subsets).

The 1-1 map will be $\chi : P([0, 1]) \rightarrow \mathcal{F} : S \mapsto \chi_S$ where this function χ_S is defined by:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

(this is called “the characteristic function of the set S ” throughout mathematics.) Now if $A, B \in P([0, 1])$ are unequal sets, then there must be at least one $x \in A$ but $x \notin B$ (or the other way around – but then just switch the names of A and B). For this x , $\chi_A(x) = 1$ while $\chi_B(x) = 0$ – thus the functions $\chi_A \neq \chi_B$ and so the map χ is 1-1. $\square$

c) Why not try looking into the hint, and seeing what the IVT from Calc I and Corollary 1.4.4 from our textbook tell us?

Let’s see, this Corollary is about reals between rationals. It says that two distinct rationals always have an irrational in between them. On the other hand, the IVT says that if a continuous function takes on two distinct values, it takes on all values in between. (These theorems seem made for each other....) From this, there is a natural

Claim: The only continuous functions on the unit interval with purely rational values are the constant (rational-valued) functions.

Proof: Say $f : [0, 1] \rightarrow \mathbb{Q}$ is continuous. Suppose f takes on two distinct values, so $f(a) \neq f(b)$ for some $a, b \in [0, 1]$. By Corollary 1.4.4 from our book there exists an irrational number x between $f(a)$ and $f(b)$. And by the IVT, there exists a $c \in (a, b)$ such that $f(c) = x$. But then f would have some irrational values, and it was assumed to be rational-valued. This contradiction implies that f takes on only one value over the whole interval $[0, 1]$, so it is a constant function.

Now we know that when looking at the rational-valued continuous functions on $[0, 1]$ we are really looking at the constant functions ... with values in $\mathbb{Q}$. But this set of functions is easy to put in bijective correspondence with $\mathbb{Q}$ itself:

$$\{\mathbb{Q}\text{-valued continuous functions on } [0, 1]\} \rightarrow \mathbb{Q} : f \mapsto f(1/2)$$

(if two constant functions agree at $1/2$, they agree everywhere, so this map is 1-1; and for any rational number q there is a constant function with that value at $1/2$, so the map is onto).

Therefore the $\mathbb{Q}$ -valued continuous functions are countable, since we know $\mathbb{Q}$ to be countable.

For continuous, *real*-valued functions, the situation is a bit different. We can skip between values, the IVT-and-Corollary 1.4.4 argument has no teeth here (so what if a real-valued function takes on an irrational value?) – so it seems there may be many more functions. But the examples in part **a**) will not work at all, because they are (badly) discontinuous – so it seems there may not be all that many more functions.

It turns out that there are still enough functions to make an uncountable set. Here is one construction (about which I will not prove all the details; it will be much easier to make better examples later in the course):

We construct an injective map from $[0, 1]$ to the set of continuous, real-valued functions on $[0, 1]$ – this set is often written as $C([0, 1])$. Given $x \in [0, 1]$, let the binary expansion of x be $.b_1b_2b_3\dots$. Now, for all $n \in \mathbb{N}$, put the point $(1/n, b_n/n)$ into the unit square in $\mathbb{R}^2$, and, if $n > 1$, connect that point to the previous one $(1/(n-1), b_{n-1}/(n-1))$ with a line segment. It can be shown that the resulting figure is actually the graph of a function f_x , which in fact becomes a continuous, real-valued function on $[0, 1]$ once we merely add the last value by defining $f_x(0) = 0$. Furthermore, it can be shown that this map $x \mapsto f_x$ is 1-1, so its image will be an uncountable subset of $C([0, 1])$, which implies, as before that $C([0, 1])$ itself is uncountable. $\square$

4. Show that if (x_n) is a sequence which converges to $x \in \mathbb{R}$, then the sequence (y_n) of averages

$$y_n = \frac{x_1 + \dots + x_n}{n}$$

(these are called *Cesaro means*) also converges to x . [Hint: this is Exercise 2.3.11 from the book.]

[Let us strategize a bit before doing the formal proof.]

We want the means y_n to converge to x . What this would mean would be that for all $\epsilon > 0$, there would be an $N \in \mathbb{N}$ such that $|y_n - x| < \epsilon$ whenever $n \geq N$. Let's fiddle with that expression:

$$|y_n - x| = \left| \frac{x_1 + \dots + x_n}{n} - x \right| = \left| \frac{x_1 + \dots + x_n - nx}{n} \right| = \left| \frac{(x_1 - x) + \dots + (x_n - x)}{n} \right| \leq \frac{|x_1 - x|}{n} + \dots + \frac{|x_n - x|}{n}$$

using the triangle inequality. Note that these terms are getting small as $n \rightarrow \infty$ for two reasons: first, $\lim_{n \rightarrow \infty} x_n = x$, so the individual terms $|x_n - x|$ are getting small; second, the sequence $(x_n)_{n=1}^\infty$ is convergent and hence bounded, and therefore the individual terms $|x_n - x|$ are also bounded and when we divide by n the resulting values are getting small. The problem, however, is that as $n \rightarrow \infty$, there are more and more of these small terms, and we need the total to get small as well. Let's just try handling all this at one time:

Fix $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} x_n = x$ there exists an $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$ whenever $n \geq N_1$. For the same reason of $(x_n)_{n=1}^\infty$ being convergent, it is bounded, say $|x_n| < M \forall n \in \mathbb{N}$. Now set we let N be a natural number greater than $N_1(1 + 2(M + |x|)/\epsilon)$. If $n \geq N$ then

$$\begin{aligned} |y_n - x| &\leq \frac{|x_1 - x|}{n} + \dots + \frac{|x_n - x|}{n} \quad [\text{as above}] \\ &= \frac{|x_1 - x|}{n} + \dots + \frac{|x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x|}{n} + \dots + \frac{|x_n - x|}{n} \quad [\text{break into two pieces}] \\ &< N_1 \cdot \frac{(M + |x|)}{n} + (n - N_1) \cdot \frac{\epsilon/2}{n} \quad \begin{array}{l} \text{[count terms, triangle inequality, bound on } |x_n|, \\ \text{count terms, control on } |x_n - x| \text{ from convergence]} \end{array} \\ &< N_1 \cdot \frac{(M + |x|)}{2N_1(M + |x|)/\epsilon} + n \cdot \frac{\epsilon/2}{n} \quad \begin{array}{l} \text{[since } n \geq N > N_1(1 + 2(M + |x|)/\epsilon) \\ \text{since } N_1 > 0 \text{ so } n > n - N_1 \end{array} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and thus $\lim_{n \rightarrow \infty} y_n = x$. $\square$