

MATH 421 MIDTERM I
TAKE-HOME PART
FALL TERM 2009

INSTRUCTIONS:

- This is an *open book and notes* test, so feel free to use your textbook and class notes. However, you must work on your own. Please also do not spend hours in the library (or on the Internet) looking for solutions to these problems.
- If you get completely stuck on a problem, feel free to contact me, however, and I could possibly consider a hint appropriate.
- Exposition is **very important**: please be as clear as you can about what you are doing. Be organized. Make a rough draft, then put your more organized thoughts into a final draft.... You get the idea.
- This is **due in class on Friday, September 25th**.

DO ANY THREE OF THE FOLLOWING PROBLEMS:

1. Assume that (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Prove that the entire sequence (a_n) must converge to this a .
[This is Exercise 2.5.4. from the book.]
2. Let (a_n) be a bounded sequence and define a set

$$S = \{x \in \mathbb{R} \mid x < a_n \text{ for infinitely many terms } a_n\}.$$

Show that there exists a subsequence (a_{n_k}) of which converges to the number $s = \sup S$. [This is a direct approach to the Bolzano-Weierstrass Theorem – so don't use B-W in your proof! Note also that this is Exercise 2.5.6 from the book.]

3. Decide whether each of the following sets is countable or uncountable, and prove your claim:
 - a) The set of **algebraic numbers**, that is, numbers $x \in \mathbb{R}$ which are roots of polynomial equations $a_n x^n + \cdots + a_1 x + a_0 = 0$ with integer coefficients, (so $a_0, a_1, \dots, a_n \in \mathbb{Z}$). [Hint: think about first whether the set of such equations is countable or not, and then you may use the basic algebraic result that every polynomial of degree n has at most n real roots. Note also that this is (part of) Exercise 1.4.12 from the book.]
 - b) The set of functions from the unit interval $[0, 1] \subset \mathbb{R}$ to the rationals, $\{f : [0, 1] \rightarrow \mathbb{Q}\}$.
 - c) **EXTRA CREDIT**: What about the set of *continuous* functions on the unit interval with values in the rationals, $\{f : [0, 1] \rightarrow \mathbb{Q} \mid f \text{ is continuous}\}$? [Hint: what must be true of all such functions? Think about the Intermediate Value Theorem from calc I and Corollary 1.4.4 from the book.] What about the set of continuous real-valued functions, $\{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$?
4. Show that if (x_n) is a sequence which converges to $x \in \mathbb{R}$, then the sequence (y_n) of averages

$$y_n = \frac{x_1 + \cdots + x_n}{n}$$

(these are called *Cesaro means*) also converges to x . [Hint: this is Exercise 2.3.11 from the book.]